MAS 4301 – Definitions/Statements

Definitions.

1. An **operation** on a set A is a function from $A \times A$ into A.
2. What it means for the operation $*$ on A to be **associative**:
   for all $a, b, c \in A$, $a * (b * c) = (a * b) * c$.
3. What it means for the operation $*$ on A to be **commutative**: for all $a, b \in A$, $a * b = b * a$.
4. What it means for the operation $*$ on A to have an **identity**.
   There is an element $e \in A$ such that for all $a \in A$, $e * a = a * e = a$.
5. The definition of a **group** is a set G equipped with an operation $*$ such that i) $*$ is associative,, ii) there is an identity for $*$, say $e \in A$, and iii) for each $a \in A$ there is a $b \in A$ such that $a * b = b * a = e$.
6. What it means for the group $(G, *, e)$ to be **abelian**: the operation $*$ is commutative.
7. Given a group $(G, *, e)$ define the **cyclic subgroup generated by** g: $< g >= \{g^n : n \in \mathbb{Z}\}$.
8. What it means for the group $(G, *, e)$ to be **cyclic**. there exists some $g \in G$ such that $G =< g >$.
9. What is a **subgroup** of a group $(G, *, e)$? A nonempty subset H such that for all $a, b \in H$, $ab \in H$ and $a^{-1} \in H$.
10. What are the trivial subgroups of a group? G and $\{e\}$
11. What it means for the function $f : A \to B$ to be **injective**.
    For all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.
12. What it means for the function $f : A \to B$ to be **surjective**.
    For each $b \in B$ there is some $a \in A$ such that $f(a) = b$.
13. What it means for the function $f : A \to B$ to be **bijective**.
    A bijection is a function which is both injective and surjective.
14. What is a **permutation** on the set S? A bijection from the set S onto itself.
15. Let G be a group and $H \leq G$. A **left coset of** H is a set of the form
    $gH = \{x \in G : \exists h \in H, x = gh\} = \{gh : h \in H\}$
16. Know the following groups: $\mathbb{Z}_n$, $U(n)$, S_n, A_n, D_n, Q_8, $G \times H$.
17. What is a **transposition** in S_n? A 2-cycle.
18. An element of S_A is said to be **even** if it can be written as the product of an even number of transpositions.
19. Define the **center** of a group $Z(G)$. $Z(G) = \{g \in G : \forall x \in G, xg = gx\}$.
20. Define the **order** of an element $g \in G$. The least positive integer $n \in \mathbb{N}$ such that $g^n = e$, it such an n exists. If not, then the order is infinite.

21. What is the **index** of a subgroup? The number of left cosets of H.
22. What is a **partition** of a set A? A set of subsets of A, say $\{A_i\}$ such that i) $A_i \cap A_j = \emptyset$ when $i \neq j$, and ii) $\bigcup_i A_i = A$.
23. Know the notation gH (left coset), Hg (right coset), gHg^{-1}, $H \leq G$, $H \trianglelefteq G$.
    $gHg^{-1} = \{x \in G : \exists h \in H, x = ghg^{-1}\}$.
24. For a subgroup $H \leq G$, the **centralizer** of H in G is $C_G(H) = \{g \in G : \forall h \in H, gh = hg\}$
25. For a subgroup $H \leq G$, the **normalizer** of H in G is $N_G(H) = \{g \in G : gHg^{-1} = H\}$
26. The subgroup H of G is said to be **normal** and write $H \trianglelefteq G$ if for all $g \in G$, $gHg^{-1} = H$.
27. Let $h \in G$. The element $x \in G$ is said to be a **conjugate** of h if there exists a $g \in G$ such that $x = ghg^{-1}$.
28. The **conjugacy class** containing h is the set of all conjugates of h.
29. Let $(G, \cdot, e_g)$ and $(H, \triangle, e_H)$ be two groups. A function $\rho : G \to H$ is a called a **homomorphism** if for all $g_1, g_2 \in G$, $\rho(g_1 \cdot g_2) = \rho(g_1) \triangle \rho(g_2)$.
30. A group homomorphism $\rho : G \to H$ is called an **isomorphism** if it is a bijection.
31. An **automorphism** of G is a group isomorphism from G onto itself.
32. The set Aut(G)=$\{\phi : G \to G : \phi$ is an automorphism of $G\}$
33. Given a group homomorphism $\phi : G \to H$, the **kernel of** ϕ is $\operatorname{Ker}\phi = \{g \in G : \phi(g) = e_H\}$
34. Given a group homomorphism $\phi : G \to H$, the **range of** ϕ is
    $\phi(G) = \{h \in H : \exists g \in G, \phi(g) = h\}$
35. Suppose $|G| = p^\alpha \cdot m$ such that gcd(p,m)=1. A **Sylow p-subgroup** is
36. Let H, K be groups and $\varphi : K \to \mathrm{Aut}(H)$ a group homomorphism. The **semi-direct product operation** is defined as
37. A group G is **simple** if
38. A group G is **Hamiltonian** if
39. The $n \times n$ matrix A belongs to the general linear group $GL_n(\mathbb{F})$ if
40. The $n \times n$ matrix A belongs to the special linear group $SL_n(\mathbb{F})$ if
41. The **determinant** of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F})$ is
42. The **inverse** of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F})$ is

43. Suppose H is an abelian group. The **automorphism of inversion** is the group homomorphism $\rho : H \to H$ defined by

44. The set of **inner automorphism of** H is denoted by $\mathrm{Inn}(G)$.

Statements.

1. Cauchy's Theorem for finite abelian groups.
2. Lagrange's Theorem
3. Sylow's (First) Theorem
4. The Correspondence Theorem
5. The First Isomorphism Theorem
6. The Class Equation
7. Cayley's Theorem for finite abelian groups.
8. Theorem that characterizes when a subgroup is normal: $g_1H = g_2H$ if and only if .