**Name:** ________________________________________

Exam 2 – MAS 4301H Fall 2017

**Remark:** Throughout the test $n$ denotes a positive integer greater than 1. Also, abstractly we consider a group $(G, \cdot, e)$. Recall that $\mathbb{Z}_n$ is the set $\{0, 1, \cdots, n-1\}$ and when equipped with addition is a group $(\mathbb{Z}_n, +, 0)$.

1. Definitions and Statements:
   (a) Let $H$ be a subgroup of $G$. State what it means for $H$ to be a *normal* subgroup of $G$

   (b) Define the center of $G$
   $$Z(G) = \{ \qquad \qquad \qquad \qquad \qquad \}$$

   (c) State what it means for $G$ to be a finite $p$-group.

   (d) State the Class Equation.

   (e) State Sylow's Theorem.

2. Give an example of a non-abelian $p$-group.

3. Give an example of a group $G$ with a non-normal subgroup $H$, i.e. give an example of a non-Hamiltonian group.

4. Let $H$ be a subgroup of $G$. Prove that the normalizer $N_G(H) = \{g \in G : gHg^{-1} = H\}$ is a subgroup of $G$

5. Consider the map $\phi : \mathbb{Z}_3 \to \mathbb{Z}_3$ defined by $\phi(x) = -x$. In class, we showed that the map $x \to x^{-1}$ is a homomorphism if and only if $G$ is abelian. So, $\phi$ is a homomorphism. Clearly, $\phi$ is an automorphism of $\mathbb{Z}_3$.

   Viewing $\phi$ as a permutation on $\{0, 1, 2\}$ exhibit its permutation representation (e.g. is it the identity, a 2-cycle or a 3-cycle?). What is the order of the permutation, and thus of $\phi$ as an automorphism?

6. Let $A = \mathbb{Z}_3$ and $G = \mathbb{Z}_4$. Define a map $* : G \times A \to A$ by

$$g * h = \begin{cases} h, & \text{if } g \text{ is even} \\ -h, & \text{otherwise.} \end{cases}$$

   Fill in the action table. You may assume that this is an action of $\mathbb{Z}_4$ on $\mathbb{Z}_3$.

| $g * a$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |

   (a) What are the orbits of this action?

   (b) For $0 \in \mathbb{Z}_3$, what is the stabilizer $G_0$?

   (c) For $1 \in \mathbb{Z}_3$, what is the stabilizer $G_1$?

7. Prove that if $K \leq N_G(H)$, then $HK \leq G$.

8. Let $\phi : G \to H$ be a homomorphism. Define the kernel of $\phi$

$$\ker \phi = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

and prove that $\ker \phi \trianglelefteq G$.

9. Let $H = \mathbb{Z}_3$ and $K = \mathbb{Z}_4$, and let $G = H \times K$, the set of ordered pairs $(h, k)$ with $h \in H$ and $k \in K$ with the produce defined by

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1 + (k_1 * h_2), k_1 + k_2)$$

where * is the action defined in Problem 6. This multiplication makes $G$ into a group of order 12.

(a) Compute $< (1,0) >= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$.

(b) Compute $< (0,1) >= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$.

(c) Compute $< (1,1) >= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$.

(d) Compute $< (2,1) >= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$.

(e) Compute $< (1,2) >= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$.

(f) How many elements of order 4 are there in $G$?

(g) How many elements of order 2 are there in $G$?

(h) How many elements of order 3 are there in $G$? Is $< (1,0) >$ a normal subgroup?

(i) How many elements of order 6 are there in $G$? Is $< (1,2) >$ a normal subgroup?

(j) Compute $(2,1)(0,1)(2,1)^{-1}$. Is $< (0,3) >$ a normal subgroup?

(k) Is $G$ cyclic? abelian? Hamiltonian? solvable? simple?
(l) Draw the lattice of subgroups of $G$.
(m) Determine the distinct conjugacy classes. What is the center of $G$?