Name:______________________

MAS 4301 – Review Final Exam

Throughout, $(G, \cdot, e)$ denotes a group and $H \leq G$.

Definitions and Statements

(1) Definition of an *operation*.
(2) Definition of an *associative* operation. Definition of an *commutative* operation.
(3) Definition of an *identity* and of an *inverse*.
(4) Definition of a *subgroup*.
(5) Definition of an *abelian* group.
(6) Definition of the cyclic subgroup generated by g: $< g >$
(7) Definition a *cyclic group*.
(8) Define $Z(G)$ and $C_G(g)$.
(9) Given a group G and a subgroup H, define the left coset of H with representative g: gH.
(10) Definition of a *permutation*. Definition of an *even* permutation.
(11) Definition of the *order* of an element: $o(g)$.
(12) Let $y \in G$. A *conjugate* of y is
(13) The *conjugacy clas*s containing $g \in G$ is
(14) Let $g \in G$. Define gHg^{-1}.
(15) What is the equivalence relation that produces cosets as its equivalence classes?
(16) What does it mean for $g_1H = g_2H$
(17) What it mean for the subgroup H to be *normal*.
(18) Define $[G : H]$ the index of H in G.
(19) The dihedral group D_n.
(20) The symmetric group S_n.
(21) Supply the definition of a *group homomorphism*.
(22) Supply the definition of a *group isomorphism*.
(23) Supply the definition of a *group automorphism*.
(24) Suppose $\rho : G \to H$ is a group homomorphism. Define $\operatorname{Ker} \rho$.
(25) What is a *simple* group?
(26) State the First Isomorphism Theorem.
(27) What is a *Hamiltonian* group?

Problems

(1) Know how to take the product of cycles in S_n and come up with the cycle decomposition. Write a $\sigma \in S_n$ as a product of transposition. Find orders of elements in S_n.
(2) What are the possible orders of elements in S_n.
(3) Find orders of elements in a group by constructing the cyclic subgroup generated by an element.
(4) Determine whether for a given n, $U(n)$ is cyclic.
(5) Multiply two matrices. Find the inverse of an element in $GL_2(\mathbb{R})$ or $GL_2(\mathbb{Z}_p)$.
(6) How many generators does Z_n have?
(7) Given a matrix $A \in GL_2(\mathbb{R})$, find $< A >$.
(8) Find orders of elements in direct products.
(9) Determine whether two groups are isomorphic. If they are not give a (correct) reason why.

(10) Let $\sigma \in \mathrm{Aut}(\mathbb{Z}_7)$ be the automorphism induced by $\sigma(1) = 4$. Find $o(\sigma)$.
(11) When is the operation on cosets well-defined? So $g_1H * g_2H = (g_1g_2)H$?
(12) Find a set representatives of the distinct cosets of a given H. For example, consider $G = D_4$ and $H =< r^2 >$. Then $[G : H] = 4$. Find representatives of the 4 distinct left cosets of H.
(13) Suppose $G =< x, y|x^n = y^m = e$ and $xy = yx^k >$
  (a) Prove by induction that $x^jy = yx^{jk}$.
  (b) Write $xy^2 = y^2x^j$ for appropriate j.
  (c) Compute $< xy >$ and find the order of xy.
(14) Draw the lattice of subgroups of S_3.
(15) Draw the lattice of subgroups of D_4.
(16) Draw the lattice of subgroups of Q_8.
(17) Draw the lattice of subgroups of $\mathbb{Z}_p$.
(18) Draw the lattice of subgroups of $\mathbb{Z}_{pq}$.
(19) Draw the lattice of subgroups of $\mathbb{Z}_{p^2q}$.
(20) Draw the lattice of subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$.

Proofs

(1) Prove using induction.
(2) State and prove Lagrange's Theorem.
(3) Prove that every cyclic group is abelian.
(4) Prove that every subgroup of a cyclic group is cyclic.
(5) Let G be an abelian group. Prove that the elements of finite order form a subgroup of G.
(6) Let H be a subgroup of G and $g \in G$. Prove that gHg^{-1} is a subgroup of G.
(7) Prove that $Z(G)$ or $C_G(g)$ is a subgroup of G.
(8) Prove that addition and multiplication on Z_n is well-defined.
(9) Prove that the set of cosets of a subgroup form a partition of G.
(10) Prove the following. Let G be a cyclic group of order n and suppose that a is a generator for G. Then $a^k = e$ if and only if n divides k.
(11) Let $(G, *, e)$ be a group. Prove that the identity is unique.
(12) Let $(G, *, e)$ be a group. Prove that the inverse of an element is unique.
(13) Prove that $| < g > | = o(g)$.
(14) Prove that the intersection of subgroups is a subgroup.
(15) For a subgroup $H \leq G$ and $g \in G$, prove that $|gH| = |H|$.
(16) Prove that for every $n \in \mathbb{Z}$, $(aba^{-1})^n = ab^na^{-1}$.
(17) Prove that $(g_1 \cdot \ldots \cdot g_n)^{-1} = g_n^{-1} \cdot \ldots \cdot g_1^{-1}$.
(18) Prove that $|H| = |gHg^{-1}|$.
(19) Prove that a given subgroup is normal, e.g. $\mathrm{Ker}\,\varphi, Z(G)$.
(20) Prove that $gHg^{-1} \leq G$.
(21) Given that $\phi : G \to H$ is a homomorphism, prove that $\phi(e_G) = e_H$.
(22) Given that $\phi : G \to H$ is a homomorphism, prove that $\phi(g^{-1}) = \phi(g)^{-1}$.
(23) Given that $\phi : G \to H$ is a homomorphism, prove that $\phi(G) \leq H$.
(24) Given that $\phi : G \to H$ is a homomorphism, prove that if G is abelian, then $\phi(G)$ is abelian.
(25) Given that $\phi : G \to H$ is a homomorphism, prove that if G is cyclic, then $\phi(G)$ is cyclic.
(26) Prove that if $G/Z(G)$ is cyclic, then G is abelian.
(27) Prove that $\varphi_g : H \to gHg^{-1}$ defined by $\varphi_g(h) = ghg^{-1}$ is an isomorphism.
(28) Prove that a given map is a homomorphism (resp. isomorphism or automorphism).
(29) Prove that a composition of homomorphisms is a homomorphism.
(30) Is the map $f : G/H \to K$ well-defined? For example $\varphi : \mathbb{Z}_7 \to \mathbb{Z}_{10}$ defined by $\varphi(x) = 2x$.