Name: ____________________

MAS 4301 – Review Exam 2

Throughout, $(G, \cdot, e)$ denotes a group and $H \leq G$.

Definitions and Statements

(1) Let $y \in G$. A *conjugate* of y is
(2) The conjugacy class containing $g \in G$ is
(3) Let $g \in G$. Define gHg^{-1}.
(4) What is the equivalence relation that produces cosets as its equivalence classes?
(5) What does it mean for $g_1H = g_2H$
(6) What it mean for the subgroup H to be normal.
(7) Define $[G : H]$ the index of H in G.
(8) The dihedral group D_n
(9) Supply the definition of a *group homomorphism*.
(10) Supply the definition of a *group isomorphism*.
(11) Supply the definition of a *group automorphism*.
(12) Suppose $\rho : G \to H$ is a group homomorphism. Define $\operatorname{Ker} \rho$.
(13) What is a *simple* group?
(14) State the First Isomorphism Theorem.
(15) What is a Hamiltonian group?

Problems

(1) Suppose $G =< x, y | x^n = y^m = e \text{ and } xy = yx^k >$
  (a) Prove by induction that $x^j y = yx^{jk}$.
  (b) Compute $< xy >$ and find the order of xy.
(2) When is the operation on cosets well-defined? So $g_1H * g_2H = (g_1g_2)H$?
(3) Find a set representatives of the distinct cosets of a given H. For example, consider $G = D_4$ and $H =< r^2 >$. Then $[G : H] = 4$. Find representatives of the 4 distinct left cosets of H.
(4) Let $\sigma \in \operatorname{Aut}(\mathbb{Z}_7)$ be the automorphism induced by $\sigma(1) = 4$. Find $|\sigma|$.
(5) Find orders of elements in direct products.
(6) Determine whether two groups are isomorphic. If they are not give a (correct) reason why.
(7) Draw the lattice of subgroups of S_3.
(8) Draw the lattice of subgroups of D_4.
(9) Draw the lattice of subgroups of Q_8.
(10) Draw the lattice of subgroups of $\mathbb{Z}_p$.
(11) Draw the lattice of subgroups of $\mathbb{Z}_{pq}$.
(12) Draw the lattice of subgroups of $\mathbb{Z}_{p^2q}$.
(13) Draw the lattice of subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$.

Proofs

(1) Prove that a given subgroup is normal, e.g. $\operatorname{Ker}\varphi, Z(G)$.
(2) Prove that $gHg^{-1} \leq G$.
(3) Given that $\phi : G \to H$ is a homomorphism, prove that $\phi(e_G) = e_H$.
(4) Given that $\phi : G \to H$ is a homomorphism, prove that $\phi(g^{-1}) = \phi(g)^{-1}$.
(5) Given that $\phi : G \to H$ is a homomorphism, prove that $\phi(G) \leq H$.
(6) Given that $\phi : G \to H$ is a homomorphism, prove that if G is abelian, then $\phi(G)$ is abelian.
(7) Given that $\phi : G \to H$ is a homomorphism, prove that if G is cyclic, then $\phi(G)$ is cyclic.
(8) Prove that if $G/Z(G)$ is cyclic, then G is abelian.
(9) Prove that $\varphi_g : H \to gHg^{-1}$ defined by $\varphi_g(h) = ghg^{-1}$ is an isomorphism.
(10) Prove that a given map is a homomorphism (resp. isomorphism or automorphism).
(11) Prove that a composition of homomorphisms is a homomorphism.
(12) Is the map $f : G/H \to K$ well-defined? For example $\varphi : \mathbb{Z}_7 \to \mathbb{Z}_{10}$ defined by $\varphi(x) = 2x$.