Name: ____________________________

MAS 4301 – Exam 1

Throughout, $(G, \cdot, e)$ denotes a group. (We do not assume that G is abelian.)
I will not give credit without work or an explanation when asked for it.

Definitions State the following definitions.

(1) Let $y \in G$. A *conjugate* of y is an $x \in G$ for which there is a $g \in G$ such that $x = gyg^{-1}$.

(2) For a subgroup H of G we define the set $N_G(H) = \{g \in G : gHg^{-1} = H\}$.

(3) Supply the definition of a *group homomorphism*. A function between two groups, say $\phi : G \to H$ such that for all $g_1, g_2 \in G$, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.

(4) Suppose $\rho : G \to H$ is a group homomorphism. Define $\operatorname{Ker} \rho$.

$$\operatorname{Ker} \rho = \{g \in G : \rho(g) = e_H\}.$$

(5) State the Class Equation. Let $g_1, \dots, g_n \in G$ be representatives of the non-central conjugacy classes. Then

$$|G| = |Z(G)| + \sum_1^n [G : C_G(g_i)].$$

(6) State the First Isomorphism Theorem. Let $\rho : G \to H$ be a group homomorphism. Then i) $\rho(G) \leq H$, ii) $\operatorname{Ker} \rho \trianglelefteq H$, and $G/\operatorname{Ker} \rho \cong \rho(G)$.

Problems

(1) Draw the lattice of subgroups of S_3.

(2) Let H, K be two subgroups of G and define

$$HK = \{x \in G : \exists h \in H, \exists k \in K \text{ such that } x = hk\}.$$

(a) In S_3, let $\sigma = (1\ 2)$ and $\tau = (1\ 3)$. Set $H =< (1\ 2) >$ and $K =< (1\ 3) >$. Find all elements of HK. Is $HK \leq S_3$?
$HK = \{e, (1\,2), (1\,3), (1\,3\,2)\}$. This is not a subgroup since $(1/2/3) = (1\,3\,2)^{-1} \notin HK$. Alternatively, 4 does not divide 6.

(b) In $D_4 =< r, s | r^4 = e = s^2, rs = sr^3 >$, find the conjugacy class containing s.
Need to conjugate s by all elements of D_4. Can exclude elements in $Z(D_4)$ and $< s >$. s, $rsr^{-1} = sr^{-1}r^{-1} = sr^2 = r^2s$, $r^{-1}sr = r^{-1}r^{-1}s = r^2s$, $rssrs = r^2s$, $r^2ssr^2s = s$, $r^3ssr^3s = r^2s$. Therefore, the conjugacy class of S is $\{s, r^2s\}$.

(3) What is the total number of subgroups of $\mathbb{Z}_5 \times \mathbb{Z}_5$. $p + 1 = 6$.

(4) Let $\sigma \in \text{Aut}(\mathbb{Z}_7)$ be the automorphism induced by $\sigma(1) = 4$. Find $|\sigma|$.
So $\sigma(x) = 4x$. Then $1 \to 4 \to 2 \to 1$. It follows that $|\sigma| = 3$.

(5) Consider $G = D_4$ and $H =< r^2 >$. Then $[G : H] = 4$. Find representatives of the 4 distinct left cosets of H.
Start with $H = \{e, r^2\}$. Pick an element not in H, say s. Then $sH = \{s, sr^2\}$. Pick a 5th element, say r. Then $rH = \{r, r^3\}$. Finally, pick one of the two remaining elements, say sr and $srH = \{sr, sr^3\}$.

$$e, s, r, srH$$

Proofs Let H and G be two groups and $\rho : G \to K$ a group homomorphism.

(1) We show that $\operatorname{Ker}\rho \trianglelefteq G$.

Proof. First, we show that $\operatorname{Ker}\rho$ is a subgroup of G. To that end, since by properties of homomorphisms we know that $\rho(e_G) = e_H$. Therefore, $e_G \in \operatorname{Ker}\rho$.

Next, let $x, y \in \operatorname{Ker}\rho$. This means that $\rho(x) = e_h$ and $\rho(y) = e_H$ Then

(Add reasons)

$$\begin{aligned} \rho(xy^{-1}) &= \rho(x) \cdot \rho(y^{-1}) \quad \text{defn of hom.} \\ &= \rho(x) \cdot \rho(y)^{-1} \quad \text{property of hom.} \\ &= e_H \cdot e_H \quad \textit{substitution} \\ &= e_H \end{aligned}$$

It follows that $xy^{-1} \in \operatorname{Ker}\rho$ and so $\operatorname{Ker}\rho \leq G$.

Next, we show that $\operatorname{Ker}\rho$ is a normal subgroup of G. Let $x \in \operatorname{Ker}\rho$ and $g \in G$.

Then $\rho(gxg^{-1}) =$ Finish proof in space

$$\begin{aligned} \rho(gxg^{-1}) &= \rho(g)\rho(x)\rho(g^{-1}) \\ &= \rho(g)e_H\rho(g)^{-1} \\ &= \rho(g)\rho(g)^{-1} \\ &= e_H \end{aligned}$$

□

(2) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$. Prove, using induction, that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

[Hint: If you forget how to multiply matrices you may ask for help on that piece alone but understand you will lose a few points.]

Clearly, for $n = 1$, $A^n = A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

Next, suppose it is true for n. We show it is true for $n+1$.

$A^{n+1} = AA^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, by induction. Now we multiply two matrices. $= \begin{pmatrix} 1 \cdot 1 + 1 \cdot 0 & 1 \cdot n + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot n + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$ Therefore, by the principle of mathematical induction the stateent is true for all n.

(**Extra Credit– Do this last**) What is the size of $GL_2(\mathbb{Z}_5)$? What is the size of $SL_2(\mathbb{Z}_5)$?

$|GL_2(\mathbb{Z}_5)| = (25-1)(25-5) = 24 \cdot 20 = 480.$

$|SL_2(\mathbb{Z}_5)| = 480/4 = 120.$