MAS 4301 – Exam 1 Review

Definitions

(1) Definition of an operation.
(2) Definition of an associative operation. Definition of an commutative operation.
(3) Definition of an identity and of an inverse.
(4) Definition of a subgroup.
(5) Definition of an abelian group.
(6) Definition of the cyclic subgroup generated by g: $< g >$
(7) Definition a cyclic group.
(8) Definition of an equivalence relation.
(9) Definition of a partition.
(10) Define $Z(G)$ and $C_G(g)$.
(11) Given a group G and a subgroup H, define the left coset of H with representative g: gH.
(12) Prove using induction.
(13) Definition of a permutation.
(14) Definition of the order of an element: $o(g)$.
(15) Given $g \in G$, define a conjugate of g.

Problems

(1) You are given a group G and a subgroup H. Find **all** left and right cosets, and the elements in them.
(2) Find the center of group ($Z(G)$).
(3) Calculate cyclic subgroups.
(4) Know how to take the product of cycles in S_n and come up with the cycle decomposition. Write a $\sigma \in S_n$ as a product of transposition. Find orders of elements in S_n.
(5) Find orders of elements in a group.
(6) Prove a certain function is injective or surjective.
(7) Determine whether a certain (non-empty) subset of a group is a subgroup.... or not.
(8) Determine whether a give operation makes a set into a group.
(9) Find the order of $U(n)$.
(10) Determine whether for a given n, $U(n)$ is cyclic.
(11) Multiply two matrices. Find the inverse of an element in $GL_2(\mathbb{R})$ or $GL_2(\mathbb{Z}_p)$.
(12) Let $\mathbb{R}^+$ denote the set of strictly positive real numbers. Show that the operation $*$ on $\mathbb{R}^+$ defined by the following is associative. (You may assume that it is an operation.)

$$x * y = \frac{xy}{x + y}.$$

(13) On the set of naturals $\mathbb{N} = \{1, 2, 3, \ldots,\}$ define an operation by $a \circ b = 2ab$. Is there a $*$-identity? If so, what is it?
(14) Find the order of a specific complex number under multiplication.
(15) How many generators does Z_n have?
(16) Given a matrix $A \in GL_2(\mathbb{R})$, find $< A >$.

Proofs

(1) State and prove Lagrange's Theorem.
(2) Prove that every cyclic group is abelian.
(3) Prove that every subgroup of a cyclic group is cyclic.
(4) Let G be an abelian group. Prove that the elements of finite order form a subgroup of G.
(5) Let H be a subgroup of G and $g \in G$. Prove that gHg^{-1} is a subgroup of G.
(6) Prove that $Z(G)$ or $C_G(g)$ is a subgroup of G.
(7) Prove that addition and multiplication on Z_n is well-defined.
(8) Prove that the set of cosets of a subgroup form a partition of G.
(9) Prove the following. Let G be a cyclic group of order n and suppose that a is a generator for G. Then $a^k = e$ if and only if n divides k.
(10) Let $(G, *, e)$ be a group. Prove that the identity is unique.
(11) Let $(G, *, e)$ be a group. Prove that the inverse of an element is unique.
(12) Prove that $| < g > | = o(g)$.
(13) Prove that the intersection of subgroups is a subgroup.
(14) For a subgroup $H \leq G$ and $g \in G$, prove that $|gH| = |H|$.
(15) Prove that for every $n \in \mathbb{Z}$, $(aba^{-1})^n = ab^n a^{-1}$.
(16) Prove that $(g_1 \cdot \ldots \cdot g_n)^{-1} = g_n^{-1} \cdot \ldots \cdot g_1^{-1}$.
(17) Prove that $|H| = |gHg^{-1}|$.