MAS 4301

1. Permutation Representation

Remark 1.1. The goal is to get you to be able to understand two theorems: Cayley's Theorem and The Orbit-Stabilizer Theorem.

Let G be a group acting on the set A. Fix $g \in G$. We are going to define a permutation from the set A back to itself. We define $\varphi_g : A \to A$ by

$$\varphi_g(a) = g \cdot a.$$

We will show that φ_g is a permutation by showing that it is injective and surjective.

To show that φ_g is , we let $a, b \in A$ and suppose that $\varphi_g(a) = \varphi_g(b)$. Applying the definition

of φ_g yields

$$g \cdot a = \varphi_g(a) = \varphi_g(b) = g \cdot b$$

We act on both side by g^{-1} and get that

$$\begin{aligned} a &= e_G \cdot a && \text{(Property 2 of Group Action)} \\ &= (g^{-1}g) \cdot a && \text{(inverses)} \\ &= g^{-1}(g \cdot a) && \text{(Property 1 of Group Action)} \\ &= g^{-1}(g \cdot b) && \text{(Substitution)} \\ &= (g^{-1}g) \cdot b && \text{(Property 1 of Group Action)} \\ &= (e_G) \cdot b && \text{(inverses)} \\ &= b && \text{(Property 2 of Group Action)} \end{aligned}$$

It follows that φ_g is injective.

Next, we show that φ_g is surjective. So let $a \in A$. Here we use g^{-1} and define $a' = g^{-1} \cdot a$. Then

$$\varphi_g(a') = g \cdot (a') = g \cdot (g^{-1}) \cdot a) = (gg^{-1}) \cdot a) = e_g \cdot a = a$$

.

It follows that φ_g is a bijection from A to itself. Therefore, we can conclude that φ_g is a

permutation. Recall that the set of permutations of A is denoted by S_A.

Observe that we started with a group acting on the set A. To each g, the induced map

$\varphi_g \in S_A$. Notice that have described a map

$$\pi : G \to S_A.$$

where $\pi(g) = \varphi_g$. The map π is called the **permutation representation** of the action of G on A.

Proposition 1.2. *Let G be a group acting on the set A. The permutation representation $\pi : G \to S_A$ is a group homomorphism.*

Proof. Let $g_1, g_2 \in G$. We aim to show that $\pi(g_1 g_2) = \pi(g_1) \circ \pi(g_2)$. This makes sense because the

operation on S_A is composition.

□

Theorem 1.3. *Let G act on itself by left multiplication. Then $\operatorname{Ker} \pi = \{e_G\}$ and so G is isomorphic to a subgroup of a permutation group.*

Proof. Recall that 1_G denotes the identity permutation, that is, the map that sends each

$g \in G$ to iteself, or g. In other words, For all $g \in G$, $1_G(g) =g$.

Suppose $h \in \operatorname{Ker} \pi$. This means that

$$\pi(h) = \varphi_h = 1_G.$$

The fact that $\varphi_h = 1_G$ means that for all $g \in G$, $\varphi_h(g)$=$hg = g$. By left cancellation we

conclude that $h = e_G$.

□