MAS 4301 – HW #5

1. Let T be the group of nonsingular upper triangular 2 2 matrices with entries in $\mathbb{R}$; that is, matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$ and $ac \neq= 0$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

(a) Show that U is a subgroup of T.

Since $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U$ we know that $U \neq \emptyset$. Also, observe that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}.$$

Therefore, U is closed under products and inverses.

(b) Show that U is abelian. This is clear since

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

(c) To show that $U \trianglelefteq T$ we take an arbitrary element $u \in U$ and an arbitrary $t \in T$ and then conjugate u by t and show it is back in U. Recall that in general the inverse of an arbitrary element in $GL_2(\mathbb{R})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} a & ay+b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/a & -\frac{b}{ac} \\ 0 & 1/c \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

where $z = \frac{ay}{c}$. Thus, $tUt^{-1} \subseteq U$. Since t was arbitrary it follows that $t^{-1}ut \in U$ and so

$$u = t(t^{-1}ut)t^{-1} \in tUt^{-1}.$$

So $U \trianglelefteq T$.

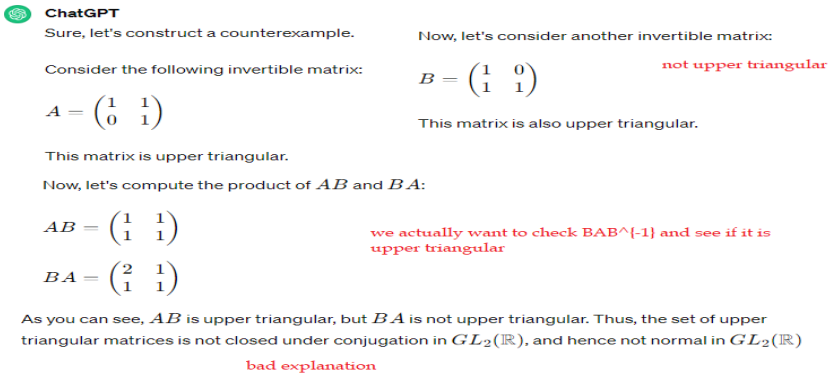
(d) Show that T/U is abelian. Let $tU, sU \in T/U$ where $t = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $s = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ are arbitrary elements in T. The goal is to show that $(ts)U = (st)U$. Recall that by Lemma 6.3 we need to show that $(st)^{-1}(ts) \in U$.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} =$$

$$\begin{pmatrix} 1/a & -\frac{b}{ac} \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1/x & -\frac{y}{xz} \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} ax & w_2 \\ 0 & cz \end{pmatrix} = \begin{pmatrix} \frac{1}{ax} & w_1 \\ 0 & \frac{1}{cz} \end{pmatrix} \begin{pmatrix} ax & w_2 \\ 0 & cz \end{pmatrix} = \begin{pmatrix} 1 & w_3 \\ 0 & 1 \end{pmatrix} \in U$$

(You can figure out what w_1, w_3 while $w_2 = ay + bz$.)

(e) Is T normal in $GL_2(\mathbb{R})$?



ChatGPT

Sure, let's construct a counterexample.

Consider the following invertible matrix:

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

This matrix is upper triangular.

Now, let's compute the product of AB and BA:

$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

As you can see, AB is upper triangular, but BA is not upper triangular. Thus, the set of upper triangular matrices is not closed under conjugation in $GL_2(\mathbb{R})$, and hence not normal in $GL_2(\mathbb{R})$

bad explanation

Now, let's consider another invertible matrix:

$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

not upper triangular

This matrix is also upper triangular.

we actually want to check BAB^{-1} and see if it is upper triangular

But we can use this to check that

$$BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \notin U$$

2. (a) The center of S_3 is $\{e\}$.
   (b) Calculate the center of $GL_2(\mathbb{R})$. The center is the set of scalar matrices. You first show that any matrix of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ commutes with all invertible matrices.
   Next, consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume it is in the center. Then
   $$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} A = A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
   . Do the math and get that $b = 0$ and $a = d$.
   Next check that
   $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A = A \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
   and get that $c = 0$.

   (c) We did this in class.

3. $U(5) = \{1, 2, 3, 4\}$. And $< 2 >= \{2, 4, 3, 1\}$ so $U(5)$ is cyclic of order 4. $U(10) = \{1, 3, 7, 9\}$. Since $< 3 >= \{3, 9, 7, 1\}$ it follows that $U(10)$ is also cyclic of order 4. So these are isomorphic. $U(12) = \{1, 5, 7, 11\}$. You can check that $5^2 = 1 = 7^2 = 11^2$ and so $U(12)$ is not cyclic of order 4. Hence not isomorphic to the others.

4. S_4 has no elements of order 6 since the only possibilities are 2-cycles, 3-cycles, 4-cycles, and 2,2-cycles. But D_{12} has an element of order 6.