MAS 4301 – HW 2

27. Let G be a group and fix $g \in G$. Define a map $\lambda_g : G \to G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation.

Proof. Since λ_g is a map from G to itself it suffices to show that λ_g is both injective and surjective. To see that λ_g is surjective we let $a_1, a_2 \in G$ for which $\lambda_g(a_1) = \lambda_g(a_2)$. Applying the function yields that $ga_1 = ga_2$ and so by left cancellation in a group, $a_1 = a_2$. We conclude that λ_g is injective.

In order to show that λ_g is surjective we take an arbitrary $a \in G$ and show there is some $x \in G$ such that $\lambda_g(a)$. Having already done the work we observe that

$$\lambda_g(g^{-1}a) = g(g^{-1}a) = a.$$

Therefore, λ_g is surjective. $\square$

Let G be a group and $H \leq G$. Fix $g \in G$ and define a map $\rho : H \to gH$ by $\rho(h) = gh$. Show that ρ is a bijection.

Proof. By now, we know that we need to show ρ is both an injection and surjection. Suppose that $\rho(h_1) = \rho(h_2)$. Then applying the function yields $gh_1 = gh_2$ and so by left cancellation $h_1 = h)2$. Thus, ρ is injective.

Next, take $x \in gH$. By definition, there is some $h \in H$ such that $x = gh$. Then $\rho(h) = gh = x$ and we conclude that ρ is surjective. $\square$

16. Find the group of rigid motions of a tetrahedron. Show that this is the same group as A_4.

Proof. Let G be the group of rigid motions of the regular tetrahedron.

Label the vertices of a regular tetrahedron by $1, 2, 3, 4$. There are a total of 6 edges. $\overline{12}, \overline{13}, \overline{14}, \overline{23}, \overline{24}, \overline{34}$. Observe that we can view a rigid motion as sending a fixed edge to any of the other edges. And you can send that fixed edge to itself in two ways by looking at the endpoints of the edge. Thus, the group of rigid motions of a tetrahedron has size 12: $|G| = 12$.

To each vertex i, we let c_i be the center of the equilateral triangle generated by the remaining three vertices. By fixing a vertex a, we produce two non-identity rotations of the triangle with vertices b, c, d. Viewing the rigid motions as a permutation of its vertices we have that G contains each of the following elements

$$(1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)$$

Notice that (1 2 3)(1 2 4) = (1 3)(2 4). Similarly, (1 3 4)(1 2 4) = (1 2)(3 4) and (1 2 3)(1 4 3) = (1 4)(2 3).

And so viewing the elements of G as permutations of $1, 2, 3, 4$, G is a group that contains a copy of A_4 but since $|G| = 12 = |A_4|$ it follows that G is a copy of A_4. □

$G = QD_8 =< x, y|x^8 = y^2 = 1, xy = yx^3 >$.

Notice that we can write each element $x^k y$ in the form yx^j.

$$x^2y = x(xy) = x(yx^3) = (xy)(x^3) = yx^3x^3 = yx^6.$$

(1) $x^3y = yx$
(2) $x^4y = yx^4$
(3) $x^5y = yx^7$
(4) $x^6y = yx^2$
(5) $x^7y = yx^5$

Since $x^4y = yx^4$ it follows that $x^4 \in Z(G)$.

Next, we calculate cyclic subgroups: Clearly, we have $< x >= \{x, x^2, x^3, x^4, x^5, x^6, x^7, e\}$. We know that this also generated by the powers which are relatively prime to 8. So $< x >=< x^3 >=< x^5 >=< x^7 >$. Also, we know that $< y >= \{y, e\}$. Next, $< x^2 >= \{x^2, x^4, x^6, e\} =< x^6 >$. And $< x^4 >= \{x^4, e\}$.

Let's look at the powers of xy.

$$(xy)^2 = (xy)(xy) = (xy)(yx^3) = x(y^2)(x^3) == xex^3 = x^4.$$

Observe that $(xy)^3 = x^4(xy) = x^5y$. Next,

$$(x^2y)^2 = (x^2y)(x^2y) = (x^2y)(yx^6) = e.$$

$$(x^3y)^2 = (x^3y)(yx) = x^4$$

(1) $< xy >= \{xy, x^4, x^5y, e\} =< x^5y >$
(2) $< x^2y >= \{x^2y, e\}$
(3) $< x^3y >= \{x^3y, x^4, x^7y, e\} =< x^7y >$
(4) $< x^4y >= \{x^4y, e\}$
(5) $< x^6y >= \{x^6y, e\}$

And that is all of the cyclic subgroups.

$\lvert g \rvert$	1	2	4	8
g	e	y, x^4 x^2y, x^4y, x^6y	x^2, x^6, xy x^3y, x^5y, x^7y	x, x^3, x^5, x^7

Since $x^4 \in Z(G)$ it commutes with any thing so take any other element of order 2 and you can generate a subgroup of order 4: $\{e, x^4, y, x^4y\}$, $\{e, x^4, x^2y, x^6y\}$.