MAS 4301

1. Group Actions

Definition 1.1. Let A be a set and G a group. We consider a function $f : G \times A \to A$. We write $g \cdot a$ instead of $f((g, a))$. Why are there two parenthesis? Because we need $(g, a) \in G \times A$.

We say that G **acts on** A if the map $g \cdot a$ satisfies the following two conditions:

(1) For all $g_1, g_2 \in G$, for all $a \in A$, $(g_1 g_2) \cdot a = (g_1) \cdot (g_2 \cdot a)$.
(2) For all $a \in A$, $e_G \cdot a = a$.

Example 1.2. (1) G acts on itself by left multiplication. Why?

Proof. For all $g_1, g_2 \in G$ and $a \in G$,

$$(g_1 g_2) \cdot a = (g_1 g_2)a = g_1(g_2 a) = g_1 \cdot (g_2 a) = g_1 \cdot (g_2 \cdot a)$$

and

$$e_G \cdot a = e_g a = a.$$

$\square$

(2) For any set A and any group G, we can define the action to be $g \cdot a = a$ for all $g \in G$ and for all $a \in A$. Is this an action? Check both steps. This action is called the trivial action.

Yes. $a = (g_1 g_2) \cdot a = g_1 \cdot a = g_1 \cdot (g_2 \cdot a)$ and $e_g \cdot a = a$ by definition.

(3) Define an action of $G = \mathbb{Z}$ on $A = \mathbb{Z} \times \mathbb{Z}$ by

$$z \cdot (m, n) = (m + zn, n).$$

Check both conditions.

For all $z, w \in \mathbb{Z}$ and all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$

$$(z + w) \cdot (m, n) = (m + (z + w)n, n) = (m + zn + wn, n)$$

while

$$z \cdot (w \cdot (m, n)) = z \cdot (m + wn, n) = (m + wn + zn, n).$$

Also,

$$0 \cdot (m, n) = (m + 0n, n) = (m, n).$$

Definition 1.3. Suppose G is acting on the set A. We fix an element $a \in A$ and define the **orbit of** a as

$$\mathrm{Orb}(a) = \{x \in A : \exists g \in G, g \cdot a = x\}.$$

Observe that $a \in \mathrm{Orb}(a)$ Why? Because $e_G \cdot a = a$.

Observe that if $b \in \mathrm{Orb}(a)$, then $a \in \mathrm{Orb}(b)$. Pf: Suppose $b \in \mathrm{Orb}(a)$. Then by definition of orbit, There is a $g \in G$ such that $g \cdot a = b$. Then

$$\begin{aligned} g^{-1} \cdot b &= g^{-1} \cdot (\underline{g \cdot a}) && \text{(Substitution)} \\ &= (g^{-1} g) \cdot a && \underline{\text{(Property of action)}} \\ &= 1 \cdot a && \underline{\text{(Cancellation)}} \\ &= a && \underline{\text{(Property of action)}} \end{aligned}$$

This shows that $\underline{a \in \mathrm{Orb}(b)}$.

Next, we show that if $c \in \mathrm{Orb}(a) \cap \mathrm{Orb}(b)$, then $\mathrm{Orb}(a) = \mathrm{Orb}(b)$. We can then conclude that the collection of orbits of A forms a partition of A.

Pf: Since $c \in \mathrm{Orb}(a) \cap \mathrm{Orb}(b)$, there exists $g_1 \in G$ such that $\underline{g_1 \cdot a = c}$, and there exists $g_2 \in G$ such that $\underline{g_2 \cdot b = c}$. In order to show that $\mathrm{Orb}(a) \subseteq \mathrm{Orb}(b)$, we take $x \in \underline{\mathrm{Orb}(a)}$ and demonstrate that $\underline{x \in \mathrm{Orb}(b)}$.

Now, by what we showed above since $x \in \mathrm{Orb}(a)$, $a \in \mathrm{Orb}(x)$. By definition of an orbit, there exists a $g \in G$ such that $g \cdot a = x$. By what we showed above since $c \in \mathrm{Orb}(a)$, then $a \in \underline{\mathrm{Orb}(c)}$ and so there exists $h \in G$ such that $h \cdot c = a$

Explain why the following is true. Substitution and properties of actions.

$$x = g \cdot a = g \cdot (h \cdot c) = g \cdot (h \cdot (g_2 \cdot b) = (ghg_2) \cdot b.$$

Thus, $x \in \mathrm{Orb}(b)$.

Do we need to show that $\mathrm{Orb}(b) \subseteq \mathrm{Orb}(a)$ or does the proof we just did show it by simply switching every instance of a by b, and vice-versa?

Example 1.4. For any group G, G acts on itself by conjugation: $g \cdot a = gag^{-1}$. Prove this is indeed an action and identify the orbit of a.

Let $g_1, g_2 \in G$ and $h \in G$. Then

$$\begin{aligned} g_1 \cdot (g_2 \cdot h) &= g_1 \cdot (g_2 h g_2^{-1}) \\ &= g_1 (g_2 h g_2^{-1}) g_2^{-1} \\ &= (g_1 g_2) h (g_2^{-1} g_1^{-1} \\ &= (g_1 g_2) h (g_1 g_2)^{-1} \\ &= (g_1 g_2) \cdot h \end{aligned}$$

Also, $e_G \cdot h = e_G h e_G^{-1} = h$.

The orbit of a is the set $\{x \in G : \exists g \in G, x = gag^{-1}\}$. In other words, the set of conjugates of a, otherwise known as the conjugacy class of a.