4/9

Exam 2 - next Tuesday

Same format as Ex 1

Def 1-5

$\mathbb{Z}_p \times \mathbb{Z}_p$

problems # 4

Ker

cosets

problem # 2

induction

normality

$G = \langle x, y \rangle$

abstract

6 or 7 Definition

True/false

Problems

Proofs

Def.: 1) a conjugate of $y$ is an element of the form $gyg^{-1}$ for some $g \in G$.

$x \sim y$ if $y$ is a conjugate of $x$

2) The conjugacy class containing $y$ is the set $\{x \in G \mid \exists g \in G \;\; x = gyg^{-1}\}$ the

3) $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

Ex. $S_3$

$H = \langle (12) \rangle = \{e, (12)\}$

$g = (13)$

$(13)(12)$

$\boxed{gHg^{-1}} =$

$(13)\, e\, (13)^{-1} = e$

$(13)(12)(13)^{-1}$

$= (123)(13) = (23)$

$(13)H(13)^{-1} = \{e, (23)\} = \langle (23) \rangle$

H is not normal

(2) $\underline{H} \le G$

(4) $g \sim x$ if $x^{-1}g \in H$. the equiv. relation ✓

$[x] = xH$.

(5) $g_1H = g_2H$ iff $g_2^{-1}g_1 \in H$.

normal
$\forall g \in G$
$\underline{gH = Hg}$

$\underline{D_4}$ $H = \langle r^2 \rangle$
$= \langle r, s \mid r^4 = e = s^2,\ rs = sr^{-1} \rangle$

(1) $H = \{e, r^2\}$. pick an element not in $H$.

(2) $rH = \{r, r^3\}$ " not in $H \cup rH$

(3) $sH = \{s, sr^2\}$ " not in $H \cup rH \cup sH$

(4) $srH = \{sr, sr^3\} = sr^3H$

---

$D_8$
16 elem

$\langle r^2 \rangle = \{e, r^2, r^4, r^6\}$

$r\langle r^2 \rangle = \{r, r^3, r^5, r^7\} = r^3\langle r^2 \rangle$

$s\langle r^2 \rangle = \{s, sr^2, sr^4, sr^6\} = sr^2\langle r^2 \rangle$

$sr\langle r^2 \rangle = \{sr, sr^3, sr^5, sr^7\}$

$r\langle r^2 \rangle r^3\langle r^2 \rangle$
$= \cancel{r^4\langle r^2 \rangle}$
$r^3\langle r^2 \rangle$

$r^3sr^2$
$sr^5r^2 = sr^7$

$r\langle r^2 \rangle \cdot s\langle r^2 \rangle$ $\quad r^3sr^2\langle r^2 \rangle$

$rs\langle r^2 \rangle$ compare.

(3)

$\varphi \in Aut(\mathbb{Z}_p)$ $\quad$ p prime

$G = \langle g \rangle$.

$\varphi \in Aut(G)$

$\varphi$ has to take a generator to a gen.

$\varphi(g)$

$\varphi(g)$ $\quad$ mult.

$\varphi(g^i) = \varphi(g)^i$

$\varphi \in Aut(\mathbb{Z}_7)$ $\quad$ add.

operation composition

$\varphi(1) = 4$.

$\varphi(4) = \varphi(1+1+1+1)$

$= 4\varphi(1) = 4 \cdot 4 = \underline{2}$.

16 mod 7 = 2

$\varphi$

$0 \to 0$

$1 \to 4$

$2 \to 1$

$3 \to 5$

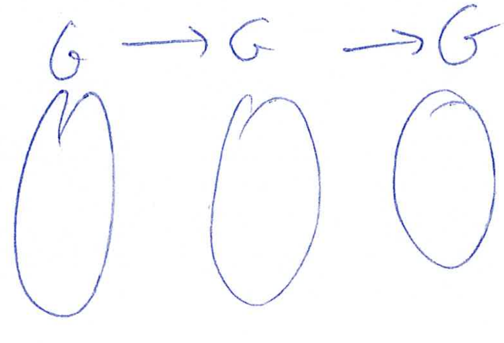
$4 \to 2$

$5 \to 6$

$6 \to 3$

$\varphi^2(1) = \varphi(\varphi(1)) = \varphi(4) = 2$

$(\varphi \circ \varphi)(1)$




$\varphi^3(1) = \varphi(\varphi^2(1)) = \varphi(2) = 1$ $\quad$ $\underline{(f \circ g)(a) = f(g(a))}$

$\varphi^3(2) = \varphi^3(1+1) = \varphi^3(1) + \varphi^3(1) = \cancel{\varphi^2}\, 1+1 = 2$

$\underline{\varphi^3 = id.}$ $\qquad$ $o(\varphi) = 3$

④

$\varphi \in Aut(\mathbb{Z}_5)$ $\quad \varphi(1) = 4.$

$0 \to 0$

$\varphi \quad 1 \to 4$ $\qquad \varphi^2(1) = \varphi(\varphi(1)) = \varphi(4) = 1$

$2 \to 3$

$3 \to 2$ $\qquad o(\varphi) = 2.$

$4 \to 1$

$\langle \varphi \rangle = \{\varphi, \varphi^2, \dots, \varphi^k = id\}$

$Aut(\mathbb{Z}_8)$

$\varphi(1) = 3$

$\varphi^2(1) = 1$

5

---

$\varphi: G \to H$ $\qquad Ker\,\varphi = \{x \in G \mid \varphi(x) = e_H\}.$

---

$\varphi(\mathbb{Z}_8) = \langle 5 \rangle$

Ex $\quad \varphi: \mathbb{Z}_8 \to \mathbb{Z}_{10}.$ $\quad \varphi(1) = 5$ $\quad \varphi(2) = 0$

$\varphi$ is a gp hom

$Ker\,\varphi = \{0, 2, 4, 6\}.$

$= \langle 2 \rangle$

$\varphi(3) = 5$

$\varphi(5) = 5$

$\varphi(7) = 5$

1st isom theorem $\quad \varphi: G \to H$ gp-hom

1) $\varphi(G) \le H.$

2) $Ker\,\varphi \trianglelefteq G$

3) $G/Ker\,\varphi \cong \varphi(G)$

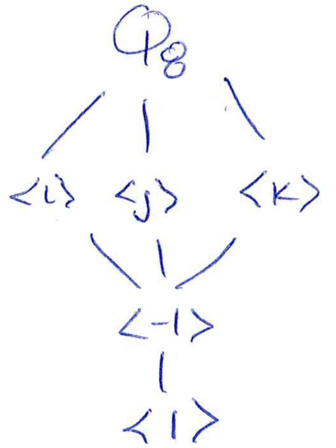
$|\varphi(G)| = |G/Ker\,\varphi| = \frac{|G|}{|Ker\,\varphi|}$

so $|\varphi(G)| \mid |G|.$

(5)

$G$ is Hamiltonian every subgroup is normal

Ex abelian, $Q_8$. $\qquad ghg^{-1} = gg^{-1} = G$




is $D_4$ Hamiltonian

$\langle r \rangle \trianglelefteq D_4$

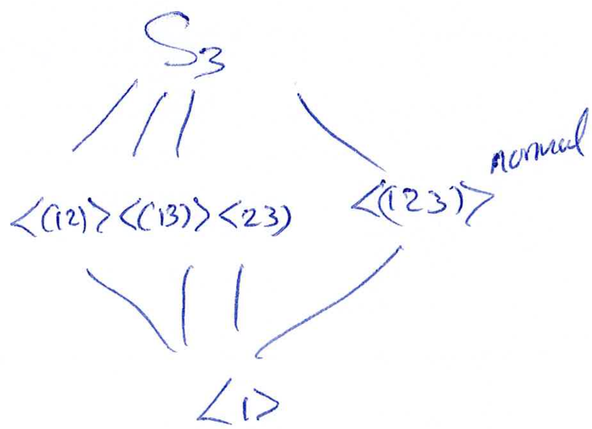
$\langle s \rangle \ntrianglelefteq D_4$

$s \qquad \langle s \rangle = \{e, s\}$

$r\langle s \rangle r^3 = \{e, sr^2\}$

$rsr^3 = sr^2$

$H \cong gHg^{-1}$




---

Exam 2 $\qquad$ 6-7 Def.

problems.

- compute cosets.
- is ~~somet~~ a subgrp normal
- draw lattice of subgps.
- ?? $\varphi \in Aut(\mathbb{Z}_p)$ $o(\varphi)$
- * $G = \langle x, y \mid x^n = e = y^m \;\; xy = yx^k \rangle$

$o(xy) \qquad \langle x \rangle \trianglelefteq G$

?? $G \times H$ $o((x, y))$

?? compute $g_1H * g_2H$. Compare $g_1g_2H$?

?? - might

* - Def.

⑥ $\underline{\text{Ker}(G), \overset{\text{normal}}{Z(G)}}$ $\varphi: G \to H$

$gHg^{-1} \leq G$

$H \trianglelefteq G$ iff

$\forall g \in G$

$\underline{gHg^{-1} \subseteq H}$

i) $\varphi(e_G) = e_H$ so $e_G \in \text{Ker}\,\varphi$.

$x, y \in \text{Ker}\,\varphi$ wts $xy^{-1} \in \text{Ker}\,\varphi$.

$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\,\varphi(y)^{-1} = e_H \cdot e_H^{-1} = e_H$. ✓

$xy^{-1} \in \text{Ker}\,\varphi$

$\underline{\text{normal}}$ $g \in G$.

Let $x \in \text{Ker}\,\varphi$ show $gxg^{-1} \in \text{Ker}\,\varphi$ $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$

$= \varphi(g)\, e_H\, \varphi(g)^{-1} = e_H$

$\text{Ker}\,\varphi \trianglelefteq G$.

---

$Z(G) = \{x \in G \mid \forall g \in G \;\; xg = gx\}$

$eg = ge \;\forall g$ so $e \in Z(G)$

$g_1, g_2 \in Z(G)$ $(g_1 g_2)g \overset{\text{assoc.}}{=} g_1(g_2 g) \overset{g_2 \in Z(G)}{=} g_1(g g_2) \overset{g_1 \in Z(G)}{=} g(g_1 g_2)$

$g_1^{-1} g$ we know $g_1 g = g g_1$ mult. left by $g_1^{-1}$

$g = g_1^{-1} g g_1$ mult on right by $g_1^{-1}$

$\underline{g g_1^{-1} = g_1^{-1} g}$. ✓

$x \in G$

$\underline{x g_1 x^{-1} h} = x x^{-1} h g_1$

$= h g_1 = h x g_1 x^{-1}$ ✓

(9)

(8) $G/Z(G)$ cyclic $\Rightarrow$ $G$ is abelian

will be on exam

By hyp. cyclic means it has a gen.

Let $x, y \in G$ WTS $xy = yx$

$G/Z(G) = \langle gZ(G) \rangle$ for some $g \in G$

$x \in$

$xZ(G) = g^i Z(G)$ , $yZ(G) = g^k Z(G)$

$x = g^i z_1$ $z_1 \in Z(G)$ $y = g^k z_2$ $z_2 \in Z(G)$

$xy = \ldots\ldots = yx$

---

$\varphi : G \to H$ hom $G$ cyclic $\Rightarrow$ $\varphi(G)$ is cyclic

$G = \langle g \rangle$ Show $\varphi(G) = \langle \varphi(g) \rangle$ 1/2 credit

$\varphi(G) = \{h \in H \mid \exists t \in G \ \varphi(t) = h\}$

$\supseteq$ $\varphi(g)^k = \varphi(g^k) \in \varphi(G)$

$h \in \varphi(G)$ $h = \varphi(x)$ some $x \in G$ By hyp $x = g^i$

$h = \varphi(x) = \varphi(g^i) = \varphi(g)^i \in \langle \varphi(g) \rangle$

$G$ abelian $\Rightarrow$ $\varphi(G)$ abelian

$h_1 h_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) = \varphi(g_2 g_1)$

$= \varphi(g_2)\varphi(g_1) = h_2 h_1$