3/26

Exam 2 Apr. $14^{th}$ (Tuesday).

- $1^{st}$ Isom
- homom. of cyclic groups
- ring theory. ($\mathbb{Z}$.)

$1^{st}$ Isom Theorem : $\varphi: G \to H$ is a group homomorphism

$(G, *, e_G)$, $(H, \Delta, e_H)$

1) $\varphi(G) \leq H$.

2) $\operatorname{Ker}\varphi \trianglelefteq G$

3) $G/\operatorname{Ker}\varphi \cong \varphi(G)$.

$\varphi(G) = \{h \in H \mid \exists x \in G \ \varphi(x) = h\}$

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Pf: 1) We know that $e_H = \varphi(e_G) \in \varphi(G)$.

Let $x, y \in \varphi(G)$ $\exists g_1, g_2 \in G$ $\varphi(g_1) = x$, $\varphi(g_2) = y$.

$$\varphi(g_1 * g_2^{-1}) = \varphi(g_1) \Delta \varphi(g_2^{-1}) \quad \text{b/c } \varphi \text{ is hom}$$
$$= \varphi(g_1) \Delta \varphi(g_2)^{-1}$$
$$= x \Delta y^{-1} = \varphi(g_1 * g_2^{-1}) \in \varphi(G).$$

one step. Subgrp Criterion. $\boxed{\equiv}$

2) we proved this b/4.

②

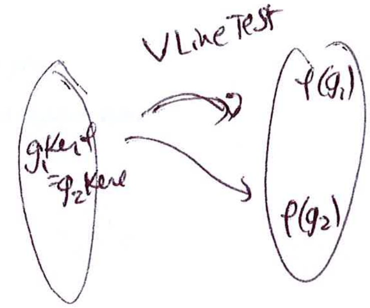
<u>Define</u> $\bar{\varphi} : G/\text{Ker}\,\varphi \to \varphi(G)$.

$gH$

$\text{Ker}\,\varphi = \{x \in G \mid \varphi(x) = e_H\}$

$\bar{\varphi}(g\,\text{Ker}\,\varphi) = \varphi(g)$.

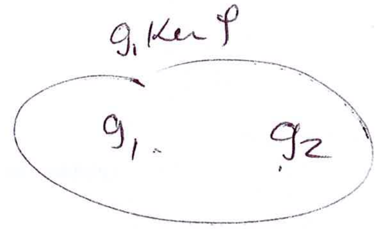
Have to show <u>well-defined</u>.




$g_1 \sim g_2$ iff $g_2^{-1}g_1 \in \text{Ker}\,\varphi$

$\S$ $g_1\text{Ker}\,\varphi = g_2\text{Ker}\,\varphi$.

<u>WTS</u> $\varphi(g_1) = \varphi(g_2)$!




$\varphi(g_2^{-1}g_1) = e_H$.

$= \varphi(g_2)^{-1} \Delta \varphi(g_1) = e_H$ mult. by $\varphi(g_2)$ on left

$\varphi(g_1) = \varphi(g_2)$ $\therefore$ $\bar{\varphi}$ is well-defined.

<u>Goal</u> show $\bar{\varphi}$ is an isom. : hom, <u>inj</u>, ~~bij~~ <u>surj</u> bij.

Let ~~$g_1\text{Ker}\,\varphi, x \in \varphi$~~ $g\,\text{Ker}\,\varphi, x\,\text{Ker}\,\varphi \in G/\text{Ker}\,\varphi$.

$\bar{\varphi}(g\,\text{Ker}\,\varphi \circledast x\,\text{Ker}\,\varphi) = \bar{\varphi}(gx\,\text{Ker}\,\varphi)$ & apply def of $\bar{\varphi}$

$= \varphi(gx)$

$= \varphi(g) \Delta \varphi(x) = \bar{\varphi}(g\,\text{Ker}\,\varphi) * \bar{\varphi}(x\,\text{Ker}\,\varphi)$.

gp. hom

$\forall a, x, b, y$

$aH = xH$
$bH = yH$ why?

$axH = byH$

iff <u>$H$ is normal</u>

③

$\bar{\varphi}$ inj. $\quad$ if $\bar{\varphi}(g_1 \operatorname{Ker}\varphi) = \bar{\varphi}(g_2 \operatorname{Ker}\varphi)$ wts $g_1 \operatorname{Ker}\varphi = g_2 \operatorname{Ker}\varphi$

$$\varphi(g_1) = \varphi(g_2)$$

$$\varphi(g_2^{-1} * g_1) = \varphi(g_2)^{-1} \Delta \varphi(g_1) = e_H$$

$\uparrow$ $g_2^{-1} * g_1 \in \operatorname{Ker}\varphi. \Rightarrow g_1 \operatorname{Ker}\varphi = g_2 \operatorname{Ker}\varphi.$

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$f : A \to B$

inj $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

suj $\forall b \in B \ \exists a \in A \ f(a) = b.$

$\bar{\varphi}$ surj $\quad \bar{\varphi} : G/\operatorname{Ker}\varphi \to \varphi(G)$

Let $x \in \varphi(G) \quad x = \varphi(g) \ \exists g \in G.$

$\bar{\varphi}(g \operatorname{Ker}\varphi) = \varphi(g) = x.$ $\quad$ surj

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$\bar{\varphi}$ is an isom.

Cor Every normal subgp. is the kernel of some homomorphism

$H \trianglelefteq G \quad \varphi : G \to G/H \quad \varphi(g) = gH.$

$\operatorname{Ker}\varphi = H$

$\{$normal subgp of $G\} = \{$Kernel of hom. $\varphi : G \to \ \}$

# (external) Direct Products

Take 2 gps $(G,*)$, $(H,\Delta)$.

$G\times H = \{(g,h) \mid g\in G, h\in H\}$. $|G\times H| = |G|\cdot|H|$.

$\mathbb{Z}_2\times\mathbb{Z}_2$

$(g_1,h_1)\otimes(g_2,h_2) = (g_1 * g_2, h_1 \Delta h_2)$.

coordinatewise

$(g,h)^{-1} = (g^{-1},h^{-1})$

$(g^{-1},h^{-1})(g,h)$

$(e_G,e_H)$

$(G\times H, \otimes, (e_G,e_H))$.

$r$ rotation by $90^\circ$

Ex. $D_4\times\mathbb{Z}_3 = \{$ 24-elements.

$(r,1)^{-1} = (r^3,2)$

$o(g,h) = \mathrm{lcm}(o(g),o(h))$

$o((r,1)) = \mathrm{lcm}(4,3) = \underline{12}$

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$\underline{G\times H}$

$G_1 = \{(g,e_H) \mid g\in G\} \trianglelefteq G\times H$

$H_1 = \{(e_G,h) \mid h\in G\} \trianglelefteq G\times H$.

Pf verbally ✓

$(x,y)(g,e_H)(x,y)^{-1} = (x,y)(g,e_H)(x^{-1},y^{-1})$

$\in G\times H$

$= (xgx^{-1}, ye_Hy^{-1})$

$= (xgx^{-1}, e_H) \in G_1$

$G_1\cap H_1 = \{(e_G,e_H)\}$

(5).

Theorem If $G$ is ~~a sub~~ a group and has two subgroups $H, K \trianglelefteq G$ s.t. $H \cap K = \{e_G\}$

Then 1) $\{h \cdot k \mid h \in H, k \in K\} \leq G$.

Call this set $HK$

2) $HK \cong H \times K$.

Pf: $e_G = \underbrace{e_G}_{\in H} \cdot \underbrace{e_G}_{\in K} \in HK$.

$h_1 k_1 \in HK$, $h_2 k_2 \in HK$.

$$h_1 k_1 h_2 k_2 = h_1 k_1 h_2 k_1^{-1} k_1 k_2$$

$$= h_1 \underbrace{(k_1 h_2 k_1^{-1})}_{\in H} \underbrace{(k_1 k_2)}_{\in K}$$

$\in HK$

$\psi : HK \to H \times K$ we need to check $\psi$ is well-defined

arbitrary $x \in HK$ $x = hk$

$h \in H$
$k \in K$

$\psi(hk) = (h, k)$

$x = hk = h_1 k_1$

~~$h_1^{-1} h = k_1 k^{-1}$~~ $H \ni h_1^{-1} h = k_1 k^{-1} \in K$

$(H \cap K = \{e_G\})$

$h_1^{-1} h = e$. so $h = h_1$

$k_1 k^{-1} = e$ $k_1 = k$

6

$\mathbb{Z}$

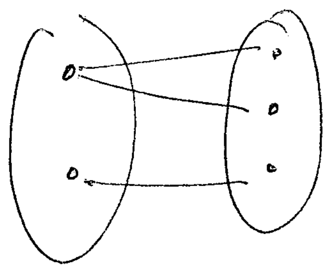
$3 \cdot 4 = 6 \cdot 2$

$3\mathbb{Z} \cap 2\mathbb{Z} = 6\mathbb{Z} \neq \{0\}$

$\boxed{H \cap K = \{e_G\}}$

$e \neq x \in H \cap K \qquad x = e \cdot x = x \cdot e.$

$\varphi(hk) = (h, k)$ check. hom
inj.
surj.

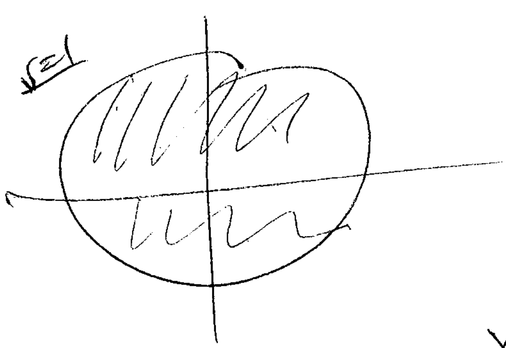


not VL well-def

$f(x) = f(y)$

$\{(x, f(x))\}$

$\{(f(x), x)\}$



$f(x) = \pm\sqrt{1 - x^2}$

$x^2 + y^2 = 1$

$f(t) = (\cos t, \sin t)$

function $\boxed{\text{well-defined}}$ $\qquad x = y \Rightarrow f(x) = f(y)$

vertical line test $\qquad gH = g'H$

7. $\mathbb{Z}/3\mathbb{Z}$

$$f : \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}.$$

$$f(x + 3\mathbb{Z}) = x.$$

$$f(2 + 3\mathbb{Z}) = 2 \qquad 2 + 3\mathbb{Z} = 5 + 3\mathbb{Z}$$

$\mathbb{Z}/3\mathbb{Z}$ $\quad f(5 + 3\mathbb{Z}) = 5 \qquad$ not well-defined

$\{\ldots, -3, 0, 3, 6, 9, \ldots\} \longrightarrow$

$\{-5, -2, 1, 4, 7, 10, \ldots\}$

$\{\ldots, -4, -1, 2, 5, 8, 11, \ldots\} \longrightarrow \begin{matrix} 2 \\ 5 \end{matrix}$

(9) Set of cont. functions of $\mathbb{R}$.

$f, g$ are cont on $\mathbb{R}$.

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\sin x \cdot e^x$

subring

set of differentiable

$x + \arctan x$

$(fg)' = f'g + g'f$

Polynomials $\mathbb{R}[x]$

$a_0 + a_1 x + \dots + a_n x^n$

mult. foil

a comm. ring w/ 1 that satisfies the property

zerodivisor property $\longrightarrow$ if $ab = 0$, then $a = 0$ or $b = 0$.

then R is called an integral domain

$\mathbb{Z}_8$ is not an integral dom $\quad 2 \cdot 4 = 0$

$x^2 = 0$ in $\mathbb{Z}_4$

8 rings

Def. $R$ has 2 operations $\cdot$ $+$

- $(R, +, 0)$ is an abelian group.
- $\cdot$ is associative.
- $\cdot$ has an identity $1$

if $\cdot$ is comm. then ~~R~~ $R$ is a comm. ring.

$\forall a, b, c \in R$ $a\cdot(b+c) = a\cdot b + a\cdot c$.

$(b+c)\cdot a = ba + c\cdot a$.

$M_2(\mathbb{R})$ ring matrix add., mult. non-comm.

$\mathbb{C}$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ - ring. $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$R_1, R_2$ $R \times S$ coord. mult.

$(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2)$

$\mathbb{C} = \mathbb{R}^2$ different mult than $\mathbb{R}\times\mathbb{R}$ coordinat

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. $\mathbb{Z}_{12}$ $4\cdot 5 = 8$