3/19

- I am creating ~~a web~~ on my webpage a folder for our files.
  I will (maybe) moving files over from Canvas to it.)
  probably by April 15th

- topics coming up.
  homomorphisms, isomorphisms,
  semi-direct + direct products.

Exam 2 - Kinds of problems

Defin — T/F proofs problems

$\downarrow$

$\langle x, y \mid \ldots \rangle$

normality $H \trianglelefteq G$

$\forall g \in G$
$gH = Hg$

$\forall g \in G$
$gHg^{-1} = H$

\varrho (var rho) ρ φ ψ

Def. A group homomorphism

Let $(G, \circ, e)$, (H, Δ, α) be groups

A function $\varphi : G \to H$ is called a group hom.

if $\forall g_1, g_2 \in G$ $\varphi(g_1 \circ g_2) = \varphi(g_1) \Delta \varphi(g_2)$.

②

Ex V, W are v.spaces

$T: V \to W$ linear transf.

$\forall x, y \in V \quad T(x+y) = T(x) + T(y)$

$\forall x \in V \ \forall c \in F \quad T(cx) = cT(x)$

lin trans. are group hom.

$T(0_V) = 0_W, \quad T(-x) = -T(x)$

$(G, \circ, e) \quad | \quad (G, \circ, e_G)$

$(H, \Delta, \alpha) \quad | \quad (H, \Delta, e_H)$

Prop If $\varphi: G \to H$ is gp hom.

1) $\varphi(e) = \alpha$.

$\varphi(e_G) = e_H$.

2) $\varphi(g^{-1}) = \varphi(g)^{-1}$

$x = x \circ x$

mult. by x^{-1}

$x^{-1} x = x^{-1} x \cdot x$

$e = e \cdot x = x$

Pf 1) $e_G = e_G \circ e_G$

$\varphi(e_G) = \varphi(e_G \circ e_G) = \varphi(e_G) \Delta \varphi(e_G)$

cancellation $e_H = \varphi(e_G)$.

$\varnothing = \varnothing \cdot ☺$

$e = ☺$

2). $\underline{\varphi(g^{-1})} \Delta \underline{\varphi(g)} \overset{\text{gphom}}{=} \varphi(g^{-1} \circ g) = \varphi(e_G) = e_H$

inverses of each other $\quad \underline{\varphi(g^{-1}) = \varphi(g)^{-1}}$

③

Let $\varphi: G \to H$ be a group homomorphis.

Define
$$\operatorname{Ker}\varphi = \{ x \in G \mid \varphi(x) = e_H \}.$$

"the <u>kernel of φ</u>" (in lin-alg. also called the nullspace.

<u>Prop</u> $\operatorname{Ker}\varphi \trianglelefteq G$

<u>Pf.</u> <u>obv.</u> $e_G \in \operatorname{Ker}\varphi$ so $\operatorname{Ker}\varphi \neq \emptyset$.

Take $x, y \in \operatorname{Ker}\varphi$. WTS $x \circ y^{-1} \in \operatorname{Ker}\varphi$.

<u>Check</u> $\varphi(x \circ y^{-1}) = \varphi(x) \Delta \varphi(y^{-1})$ φ gp hom

$= \varphi(x) \Delta \varphi(y)^{-1}$ prop. of gphom.

$= e_H \Delta e_H^{-1}$ $y, x \in \operatorname{Ker}\varphi$

$= e_H.$ So it's subgp

$\operatorname{Ker}\varphi \leqslant G$.

WTS $\forall g \in G$ $g \operatorname{Ker}\varphi g^{-1} = \operatorname{Ker}\varphi$.

<u>Let $g \in G$.</u> let $x \in g \operatorname{Ker}\varphi g^{-1}$ this means $x = gkg^{-1}$ for some $k \in \operatorname{Ker}\varphi$.

$\varphi(x) = \varphi(g \circ k \circ g^{-1}) = \varphi(g) \Delta \varphi(k) \Delta \varphi(g)^{-1}$

$= \varphi(g) \Delta e_H \Delta \varphi(g)^{-1} = \varphi(g) \Delta \varphi(g)^{-1}$

$= e_H.$

$x \in \operatorname{Ker}\varphi$. so $\forall g \in G$ $g \operatorname{Ker}\varphi g^{-1} \subseteq \operatorname{Ker}\varphi$.

④ if you show $\forall g \in G$ $gHg^{-1} \subseteq H$. then $H \trianglelefteq G$

(Shortcut)

$(\Downarrow \; \forall g \in G \; gHg^{-1} = H.$

Let $g \in G$. Let $h \in H$. what do you know about

Let WTS $H \subseteq gHg^{-1}$ $g^{-1}Hg \subseteq H$..

so $g^{-1}hg \in H$

$h = g\underbrace{(g^{-1}hg)}_{\in H}g^{-1} \in gHg^{-1}$ so $H \subseteq gHg^{-1}$

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ask you to show some specific $H \trianglelefteq G$.

Let $g \in G$. WTS $gHg^{-1} = H$.

give a proof $x \in gHg^{-1}$ - - - a miracle happens . $x \in H$.

(6) φ "G, H are isom". $G \cong H$

φ $G \cong H$.

- if G is cyclic, then H is cyclic
- if G is abelian, then H is abelian
- $|G| = |H|$ (bijection)
- for $g \in G$, if $o(g) = n$, then $o(\varphi(g)) = n$.
- the lattice of subgps of $G + H$ looks alike.
- if G has k-many elements of order n, then so does H.
- $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$ not cyclic $D_4 \not\cong Q_8$ – 6 elements of order 4

  ↓ 2 elements of order 4.

Take a small number n.

"Classify all groups of order n".

This means give a list of groups, all of order n, no two are isomorphic, and any group of order n is isom. to one in the group

(5)

Def. An ~~isomorphism~~ group isomorphism is a group hom. $\varphi: G \to H$ such that φ is a bijection

Two groups G, H are called isomorphic if there exists a grp. isom $\varphi: G \to H$.

Question

- $\mathbb{Z}, \mathbb{R}$ are not isomorphic (T)

  $|\mathbb{Z}| < |\mathbb{R}|$ so no bijections.

- $\mathbb{Z}, \mathbb{Q}$ are isomorphic (F)

  $\mathbb{Z}$ cyclic, $\mathbb{Q}$, not cyclic.

- $\mathbb{Z}_{12}, A_4$ so not isomorphic

  abelian not abelian

- A_4, D_6

  has an element of order 6

  no element of order 6.

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$n=1$	$\{0\}$
$n=2$	$\mathbb{Z}_2$
$n=3$	$\mathbb{Z}_3$
$n=4$	$\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$
$n=5$	
$n=6$	$\mathbb{Z}_6, S_3$
$p=n$ prime	$\mathbb{Z}_p$

MAS 4301 – Generators/Relations Pt 2

1. There are only two possible ways to have a subgroup of order 4. Explain.

$\mathbb{Z}$ cyclic or generated by 2 element of order 2

$\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$

2. Let G be a group and $x, y \in G$ such that $|x| = 8$ and $|y| = 2$. The elements satisfy the relation $xy = yx^3$.
   (a) Show that $x^2y = yx^6$.
   (b) Show that $x^3y = yx$.
   (c) Continue on with the others.
   (d) Use the relations to find the cyclic subgroups generated by each of the following elements: $xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y$.

$xy = yx^3$ $\quad x^2y = yx^6$ $\quad x^7y = yx^5$

$x^3y = yx$

$x^4y = yx^4$

$x^5y = yx^7$

$x^6y = yx^2$

induction

$x^ky = yx^{3k}$

$\langle xy \rangle = \{xy, x^4, x^5y, e\} = \langle x^5y \rangle$

$\langle x^2y \rangle = \{x^2y, e\}$

$\langle x^3y \rangle = \{x^3y, x^4, x^7y, e\}$

$(xy)(xy) = x(yx)y = x(x^3y)y = x^4$

$(xy)^4 = xy\,x^5y = xx^7yy = e$

$x^2yx^2y = x^2x^6yy = e$.

$x^3yx^3y = x^3xyy = x^4$

$(x^3y)^3 = x^3y\,x^4 = x^7y$

$(x^3y)x^7y = x^3x^5y^2 =$

$\langle x^4y \rangle = \{x^4y, e\}$

$\langle x^6y \rangle = \{x^6y, e\}$

$(xy)^3 = xy(xy)^2 = xyx^4 = xx^4y = x^5y$

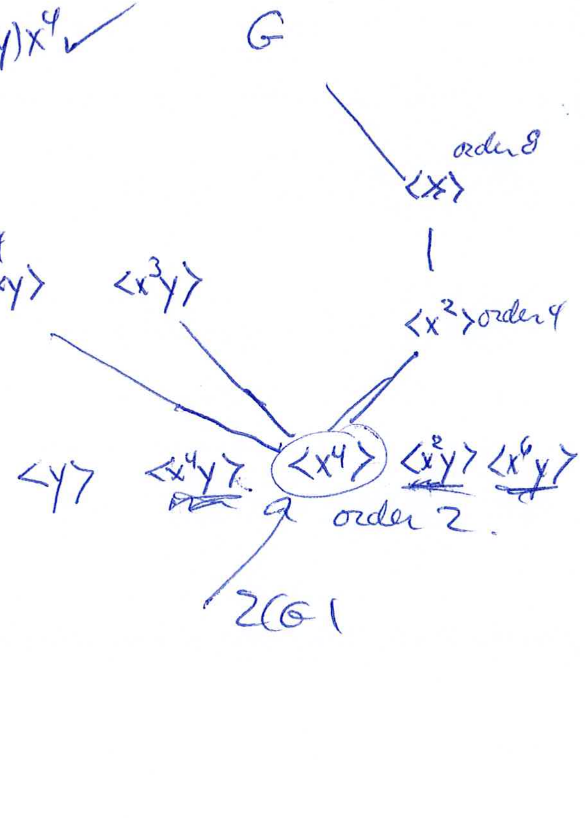
3. Show that $x^4 \in Z(G)$ (the center of G).

$x^4(x^iy^j)$ $\quad j=0$ $\quad x^4x^iy = x^ix^4y = (x^iy)x^4$

$xy = yx^3$

$x^2y = x(xy) = (xy)x^3 = yx^3x^3 = yx^6$

Show (induction) $x^ky = yx^{3k}$



$\langle x, y \mid x^8 = y^2 = e,\ xy = yx^3 \rangle$