3/17

- Welcome Back!
- Happy St. Patty's Day
- OH make-up. (email)
- HW #4.

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normal subgroups

Definition Let $H \leq G$. We say H is a normal subgroup of G if $\forall g \in G$

$$gHg^{-1} = H.$$

Facts $\forall H \leq G, \forall g \in G \quad gHg^{-1} \leq G$

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\} = \{x \in G \mid \exists h \in H \ x = ghg^{-1}\}$$

~~gHg~~

$|gHg^{-1}| = |H|$. $\quad \varphi_g : H \to gHg^{-1}$ defined by $\varphi_g(h) = ghg^{-1}$

φ_g is injective + surjective.

φ_g bijection.

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Prop $H \leq G$. then H is normal in G iff

$\forall g \in G \quad gH = Hg$ $\quad |gH| = |H|$

Ex S_3

cosets of H

$\{H, (13)H, (23)H\}$

$H = \langle (12) \rangle = \{e, (12)\}$ $\quad g = (13)$.

$gH = \{g \cdot e, g(12)\} = \{(13), (13)(12)\}$

$= \{(13), (123)\}$.

$Hg = \{\{e \cdot (13), (12)(13)\} = \{(13), (132)\}$

Does $gH = Hg$ **No** so $\langle (12) \rangle$ is not normal.

Notation $H \leq G$ "H is a subgp of G"

$H \trianglelefteq G$ "H is a normal subgp of G"

(3) <u>Ex.</u> of normal subgps

(1) $G \trianglelefteq G$

(2) $\{e\} \trianglelefteq G$

(3) $Z(G) \trianglelefteq G$. the center of G

$Z(G) = \{g \in G \mid \forall x \in G,\ gx = xg\}$

<u>Prove</u> $\forall g \in G$ $gZ(G)g^{-1} = Z(G)$

Let $g \in G$. & $x \in gZ(G)g^{-1}$ this means

$\exists z \in Z(G)$ s.t. $x = gzg^{-1} = zgg^{-1} = ze = z$.

so $x = z \in Z(G)$. so $gZ(G)g^{-1} \subseteq Z(G)$.

for the reverse inclusion, let $z \in Z(G)$

then $z = z \cdot e = zg \cdot g^{-1} = gzg^{-1} \in gZ(G)g^{-1}$

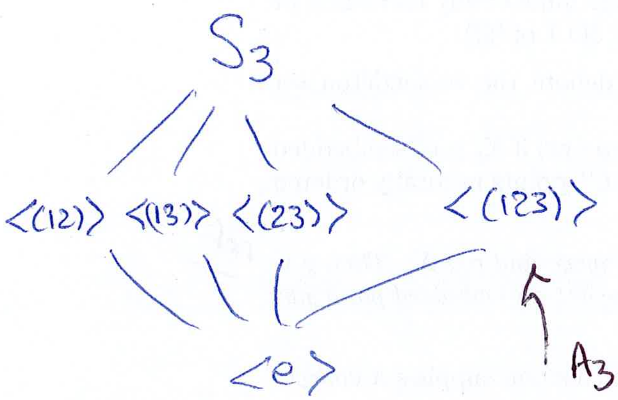
(3a) Same proof works for any $H \leq G$ that satisfies

$H \subseteq Z(G)$.

(4) ~~if~~

(4) if $H \leq G$ (finite) and H is the only subgp of G of & order $|H|$. (of a given order)

then $H \trianglelefteq G$.



$\langle(123)\rangle \trianglelefteq S_3$.

pf. of (4)

$|gHg^{-1}| = |H|$ so since H is the only subgp of that size $gHg^{-1} = H$.

(5) if $[G:H] = 2$ ($|G|$ finite) the index of H is 2. then $H \trianglelefteq G$.

↓ # of left cosets.

take H is one left coset, choose $g \in G \setminus H$.

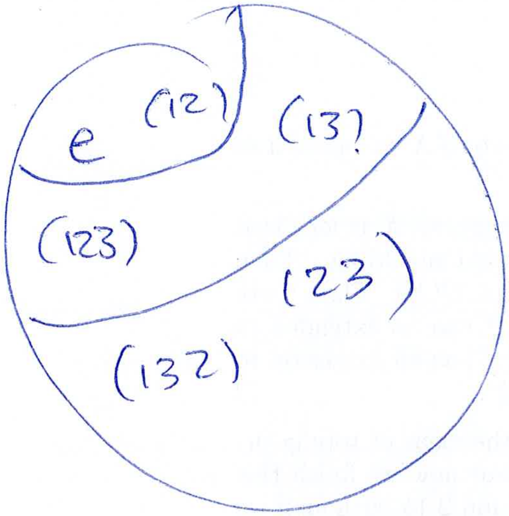
gH so $G = H \cup gH$.

right cosets H, Hg

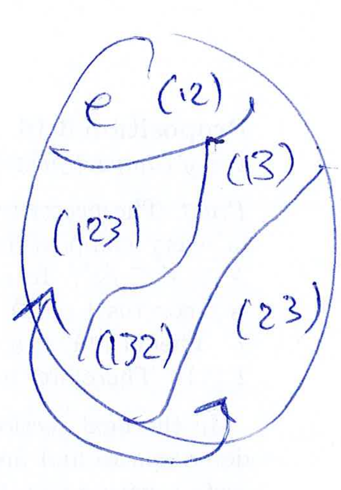
here $gH = G \setminus H$

$Hg = G \setminus H$ so $\underline{gH = Hg}$.

(5) $H = \langle (12) \rangle$ $\quad G = S_3$



left ~~costes~~ cosets



$\underline{Q_8}$ $\quad |Q_8| = 8$

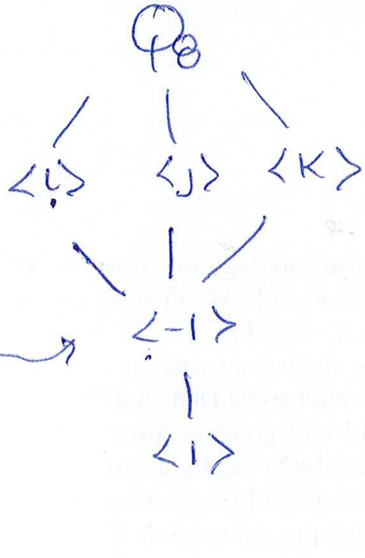
$= \{\pm 1, \pm i, \pm j, \pm k\}$

Q_8 not abelian

(if G abelian, then every subgroup is normal.)

$H = \langle i \rangle$

$jH = kH$



$Z(Q_8)$

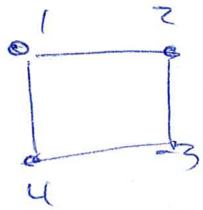
$\langle i \rangle = \{i, -1, -i, 1\}$

$[Q_8 : \langle i \rangle] = \cancel{2} = \dfrac{|Q_8|}{|\langle i \rangle|} = \dfrac{8}{4} = 2$

Def. A group for which ~~e~~ every subgroup is normal is called $\underline{\text{Hamiltonian}}$

Q_8 Hamiltonian not abelian.

(6) $D_4 = \langle r, s \mid r^4 = e = s^2 \quad srs^{-1} = r^3 \rangle$

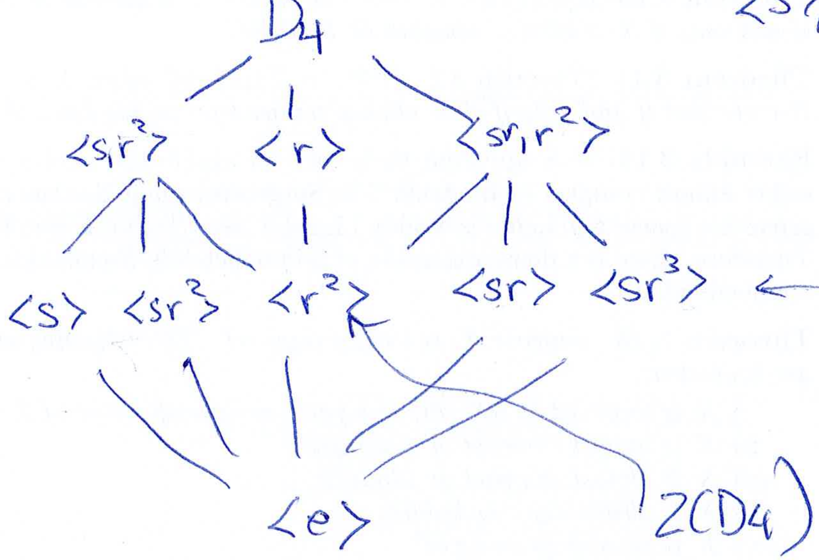


$r = (1234)$

$s = (24)$.

$\langle s \rangle, \langle sr^2 \rangle$ not normal
$\langle sr \rangle$
$\langle sr^3 \rangle$

all normal, index 2



$\langle s \rangle = \{e, s\}$

$r\langle s \rangle = \{r, rs\}$ $\quad rs \neq sr$

$\langle s \rangle r = \{r, sr\}$ not normal.

$A_n \trianglelefteq S_n$. $\quad |S_n| = n!$ $\quad |A_n| = \frac{n!}{2}$

$[S_n : A_n] = \frac{n!}{n!/2} = 2$.

~~a group is called simple~~

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A group G is called simple if the only normal subgps are G and $\{e\}$

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not simple means it has a non-trivial normal subgroup.

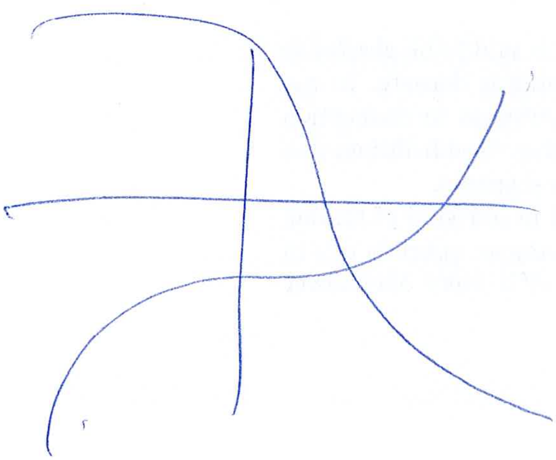
The Classification of finite simple groups

S_n, Q_8, D_n not simple. A_n $(n \geq 5)$ is simple

Group Theory original. gps of permutations of roots of polynomial

$f(x) = ax^2 + bx + c$

degree 2 QF: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$



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$f(x) \in \mathbb{Q}[x]$

solvable by radicals

roots only involve n^{th}-roots & field opns.

Gal(f(x))

solvable

a chain of normal subgps

A_5 is not solvable