① 3/5

$x^j y^k$

$G = \langle x, y \mid \boxed{x^5 = y^4 = e}.\ xy = yx^2 \rangle$ $|G| = 20$

$xy = yx^2$ super important

$x^2y = x(xy) = xyx^2$
$= (yx^2)x^2 = yx^4$

$xy = yx^2$
$x^2y = yx^4$
$x^3y = yx$
$x^4y = yx^3$
ok?

$x^3y = x(x^2y) = xyx^4$
$= yx^2x^4 = yx^6$

$\langle xy \rangle$

$(xy)^2 = xyxy$
$= x(x^3y)y$
$= x^4y^2$

$\underline{x^k y = y x^{2k}}$ $k=1$ ✓ (Basis)

§ true for k. WTS true for $k+1$

$x^{k+1}y = xx^ky = xyx^{2k} = yx^2x^{2k}$
$= yx^{2+2k} = yx^{2(k+1)}$. ☒

$(xy)^3 = \cancel{xy(}$
$\cancel{= (xy)^2 x}$
$xy(xy)^2 = xyx^4y^2$
$= xx^2yy^2 = x^3y^3$

peel off xy to the left.

$\langle e \rangle$

$\langle x \rangle = \{x, x^2, x^3, x^4, e\} = \langle x^2 \rangle$
$\langle x^3 \rangle$
$= \langle x^4 \rangle$

$(xy)^4 = xy(xy)^3$
$= xyx^3y^3$
$= x(x^4y)y^3 = e$
$= x^5y^4 = e.$

$o(xy) = 4$

$\langle xy \rangle = \{xy, x^4y^2, x^3y^3, e\} = \langle x^3y^3 \rangle$

$\langle x^4y^2 \rangle = \{x^4y^2, e\}$

$x^4y^2 = (xy)^2$

$e = (xy)^4 = (x^4y^2)^2$

$\langle x^2y \rangle$

②

5·3   5·7   2·5   p·q

not Cayley table list

$xy = yx^2$

$p^2 \cdot q$ pq ?? p^2 ??

e ✓	y ✓	y^2 ✓	y^3 ✓
x ✓	xy ✓	xy^2 ✓	xy^3 ✓
x^2 ✓	x^2y ✓	x^2y^2 ✓	x^2y^3 ✓
x^3 ✓	x^3y ✓	x^3y^2 ✓	x^3y^3 ✓
x^4 ✓	x^4y ✓	x^4y^2 ✓	x^4y^3 ✓

$\langle e\rangle, \langle x\rangle, \langle y\rangle$

$\langle xy\rangle$

$\langle x^2y\rangle = \{x^2y, x^3y^2, xy^3, e\} = \langle xy^3\rangle$

$\langle x^3y^2\rangle = \{x^3y^2, e\}$

$\langle x^3y\rangle = \{x^3y, x^2y^2, x^4y^3, e\}$

$\langle x^2y^2\rangle$

so far
the elements all have
order 1, 2, 4 or 5

$(x^4y)x^4y = x^4x^2yy = xy^2$

$x^4yxy^2 = x^4x^3yy^2 = x^2y^3$

$x^4yx^2y^3 = x^4xyy^3 = e$.

$\langle x^4y\rangle = \{x^4y, xy^2, x^2y^3, e\}$

$\langle xy^2\rangle$

$(x^2y)x^2y = x^2(yx^2)y$
$= x^2xyy$
$= x^3y^2$

$\cancel{(x^2y)(x^2y)^2 = x^2y\,x^3y^2}$
$= \cancel{x^2xyy^2} = x^3y^3$

$(x^2y)^3 = (x^2y)(x^2y)^2 = (x^2y)(x^3y^2)$
$= x^2(yx^3)y^2$
$x^2(x^4y)y^2 = xy^3$

$(x^2y)^4 = x^2y\,(x^2y)^3 = x^2y\,xy^3 = x^2(xy)y^3$
$= x^5y^4 = e$

$(x^3y)^2 = x^3yx^3y = x^3x^4yy = x^2y^2$

$x^3yx^2y^2 = x^3xyy^2 = x^4y^3$

$x^3yx^4y^3 = x^3x^2yy^3 = e$

(3) $\underline{\mathbb{Z}_4}$ Groups of order 4.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$

$\{(0,0), (0,1), (1,0), (1,1)\}$

G

Exam 2 Give you gen./rel.

$x^k y = \ldots$

$\langle xy \rangle$

$\langle y \rangle$ $\langle xy \rangle$ $\langle x^2 y \rangle$ $\langle x^3 y \rangle$ $\langle x^4 y \rangle$ $\langle x \rangle$

$\langle y^2 \rangle$ $\langle x^4 y^2 \rangle$ $\langle x^3 y^2 \rangle$ $\langle x^2 y^2 \rangle$ $\langle xy^2 \rangle$

$\langle e \rangle$

$\langle x, y^2 \rangle$ $\langle y^2$

④ $\underline{H \leq G}$

$a, b \in G$ and define

$\underline{a \sim b \text{ iff } b^{-1}a \in H}$.

refl $\quad a^{-1}a = e \in H \Rightarrow a \sim a$

$a \sim b \quad b^{-1}a \in H \Rightarrow (b^{-1}a)^{-1} \in H$

$a^{-1}b \in H \Rightarrow b \sim a$

$a \sim b,\ b \sim c$

$b^{-1}a \in H \quad c^{-1}b \in H$

$c^{-1}a = (c^{-1}b)(b^{-1}a) \in H \quad c \sim a$

$[a] = aH$

Pf: $\text{if } a \sim b$ then $b^{-1}a \in H$

$b \sim a \quad \underline{a^{-1}b} \in H$.

$b = a \cdot \underbrace{(a^{-1}b)}_{\in H} \in aH$.

$[a] \subseteq aH$.

$\overset{\in aH}{ah},\ h \in H$

$(ah)^{-1}a = h^{-1}a^{-1}a = h^{-1} \in H$.

so $a \sim ah$ so $ah \in [a]$

$aH \subseteq [a]$

$\therefore aH = [a]$

$n\mathbb{Z} \leq \mathbb{Z} \qquad \underline{n > 1}$

modular arithmetic

$a \sim b$ iff $n \mid (b-a)$

eq. relation

$[a] = \{x \mid n \mid (x-a)\}$

There are n-many equivalence classes

$[G:H]$ = the number of left cosets.

$\forall a \in H \quad \underline{|aH| = |H|}$

$|G| = [G:H] \cdot |H|$

$[G:H] = \dfrac{|G|}{|H|}$ finite case

(5)

$\mathbb{Z}/n\mathbb{Z}$ = set of left cosets. $|\mathbb{Z}/n\mathbb{Z}| = n$

$[a] \oplus [b] = [a+b]$ well-defined

gH $\cdot$

$(\mathbb{Z}/n\mathbb{Z}, \oplus, [0])$ is a group.

$g+H$ $+$

$[a]=[c]$
$[b]=[d]$ $\Rightarrow$ $\cancel{[a+b]}$ $[a] \oplus [b] = [c] \oplus [d]$

$[a+b] = [c+d]$.

$-[a] = [-a]$

G/H set of left cosets.

$(g_1 H) * (g_2 H) = (g_1 g_2) H$.

$(a+n\mathbb{Z}) \oplus (b+n\mathbb{Z})$

$(a+b)+n\mathbb{Z}$

Is it well-defined. (not in general).

?? $g_1 H = x_1 H$ and $g_2 H = x_2 H$ then $(g_1 g_2) H = (x_1 x_2) H$.

$x_1^{-1} g_1 \in H$ and $x_2^{-1} g_2 \in H$. then $(x_1 x_2)^{-1} (g_1 g_2 \in H$

$x_2^{-1} x_1^{-1} g_1 g_2 \in H$

$H \leq G$ $x_1^{-1} g_1 x_2^{-1} g_2 \in H$

if G is abelian then it is well-defined.

⑥

§ H has the property that $\forall s \in G$

$\underline{sHs^{-1} = H}$ (this is equivalent to saying that

$\underline{\forall s \in G \quad sH = Hs}$.

~~ev the~~ each left coset equals its right coset.

if H has this property, then

$\underline{x_1^{-1}g_1 \in H}, \underline{x_2^{-1}g_2 \in H} \Rightarrow \underline{x_2^{-1}x_1^{-1}g_1g_2 \in H}$.

$\cancel{g_1H = x_1H}$

so the mult.

$$\boxed{x_2^{-1}(x_1^{-1}g_1)g_2}$$

call $x_1^{-1}g_1 = h$.

$= x_2^{-1}hg_2$

$hg_2 \in Hg_2 = g_2H$.

$= \underbrace{x_2^{-1}g_2}_{\in H} h_1 \in H \in \underline{H}$

$hg_2 = g_2h_1$

$h_1 \in H$.

So $\cdot$ is well-defined on G/H.

$D_4 = \langle r, s \mid r^4 = s^2 = e,\ rs = sr^{-1} \rangle$

$rs = sr^3$

$H = \langle s \rangle = \{e, s\}$.

$rH = \{r, rs\}$. $\quad Hr = \{r, sr\}$

$rs = sr$? no.

So $rH \neq Hr$, $\quad rs = sr^3$ yes

$rsH = \{rs,$ ~~e~~

~~$rH \cdot eH$~~ $= rH$. $\quad rH$ ~~$\cdot rsH$~~

$r^3sH = \{r^3s, r^3\}$

~~rsH~~ $\quad rH \cdot r^3sH = (r^4sH = sH = e.$

~~$rsH \cdot r^3sH$~~

$rH = rsH \qquad r^3sH = r^3H$

$rH \cdot r^3sH = r^4sH = sH = H$ $\qquad$ mult. not

$rsH \cdot r^3H = rsr^3H = r(rs)H = r^2sH \neq H$ $\qquad$ well-defined

---

Need to show $\quad$ 1) $\forall s \in G,\ sHs^{-1} = H$ iff $\forall s \in G\ sH = Hs$.

2) if mult. is well-defined, then $\forall s \in G\ sH = Hs$

A subgroup w/ this property is called normal