3/30

#s get bigger as you go down

$$\begin{tikzcd}
& \mathbb{Z}_{p^2q} \arrow[dl, dash] \arrow[dr, dash] & \\
\langle p\rangle \arrow[d, dash] \arrow[dr, dash] & & \langle q\rangle \arrow[dl, dash] \\
\langle p^2\rangle \arrow[dr, dash] & \langle pq\rangle \arrow[d, dash] & \\
& \langle p^2q\rangle = 0 &
\end{tikzcd}$$

h is conj to g if

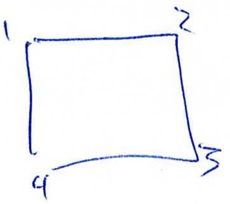
$\exists\, x \in G$

$xgx^{-1} = h$

$$D_4 = \langle r, s \mid r^4 = s^2 = e,\ rs = sr^3 \rangle$$

$s^{-1}rs = r^3$

r is conj to r^{-1}



$r = (1234)$

$s = (24)$

Question

$$r^2s = r(rs) = r(sr^3)$$
$$= (rs)r^3$$
$$= (sr^3)r^3 = sr^2 \quad \Rightarrow$$

$r^2s = sr^2$ so r^2 commutes w/ s. it also commutes w/ r } both generators

so $r^2 \in Z(D_4)$

$$r^3s = r(r^2s) = r(sr^2)$$
$$= (rs)r^2 = (sr^3r^2)$$
$$= sr^5 = sr^4r = ser = sr.$$

② computing cyclic subgroup.

$\langle e \rangle = \{e\}$ $\quad |D_4| = 8$

$\langle r \rangle = \{r, r^2, r^3, e\} = \langle r^3 \rangle$ $\quad \langle s \rangle = \{s, e\}$

$\langle r^2 \rangle = \{r^2, e\}$

$\langle rs \rangle = \{rs, e\}$

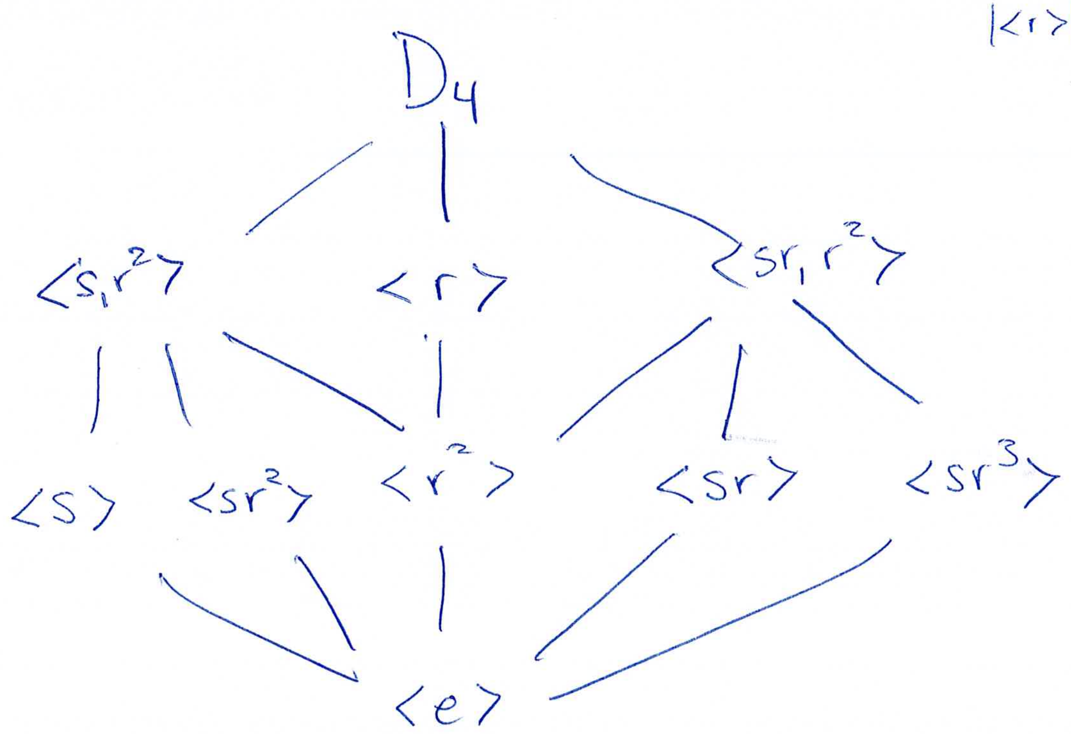
$(rs)(rs) = r(sr)s$
$= r(r^3 s)s$
$= r^4 s^2 = e$.

$\langle r^2 s \rangle = \{r^2 s, e\}$

$\langle r^3 s \rangle = \{r^3 s, e\}$ $\quad r^3 s r^3 s$
$r^3 (rs) s$

$r^2 (s r^2) s$
$= r^2 r^2 s s = e$

$|D_4| = 8$
$|\langle r \rangle| = 4$



③ $G = \langle x, y \mid x^7 = y^3 = e, \; xy = yx^2 \rangle$

$xy = yx$
$yx = yx^2$

$\langle x \rangle = \{x, x^2, \ldots, x^6, e\} = \langle x^2 \rangle = \langle x^3 \rangle = \cdots = \langle x^6 \rangle$

$y^j x^{i}, x$

$\langle y \rangle = \{y, y^2, e\} = \langle y^2 \rangle$

$\boxed{y^i x^j}$

$x^2 y = x(xy) = x(yx^2)$
$= (xy)x^2 = yx^2x^2 = yx^4$

$x^3 y = x(x^2y) = x(yx^4) = (xy)x^4 = (yx^2)x^4 = yx^6$

$x^4 y = x(x^3 y) = xyx^6 = yx^2x^6 = yx^8$

Claim $x^k y = y x^{2k}$ $\forall k \in \mathbb{N}$

Princ. Math

Pf: $S = \{k \in \mathbb{N} \mid x^k y = y x^{2k}\}$

$S \neq \emptyset$ $S \subseteq \mathbb{N}$.

S sati 2 prop

1) $1 \in S$

Show S has 2 prop. i) $1 \in S$

ii) if $n \in S$, then $n+1 \in S$.

2) if $n \in S$, then $n+1 \in S$

$S = \mathbb{N}$.

Base ~~Ind.~~ Step. $k=1$ $x^1 y = y x^2 = y x^{2 \cdot 1}$ ✓

if not, $\mathbb{N} \setminus S \neq \emptyset$

Ind. Step $\$$ it's true for k.

m is least in S.

$m = (m-1)+1$

$x^{k+1} y = x(x^k y) \overset{\text{by ind step}}{=} x(y x^{2k}) = (xy)x^{2k}$
$= yx^2x^{2k} = yx^{2+2k} = yx^{2(k+1)}$

So it's true for $k+1$.

Conc by Princ. Math induction $x^k = yx^{2k}$.

(4)

$x^k y = y x^{2k}$

$|G| = 21$

$o(g) = 1, 3, 7,$

$$xy = yx^2$$
$$x^2y = yx^4$$
$$x^3y = yx^6$$
$$x^4y = yx$$
$$x^5y = yx^3$$
$$x^6y = yx^5$$

any $H \leq G$

$|H| = 1, 3, 7, 21$ in G

all cyclic.

$\langle x \rangle$

$\langle xy \rangle$
$= \{xy, x^5y^2, e\} = \langle x^5y^2 \rangle$

$\langle x^2y \rangle = \{xy, x^3y^2, e\} = \langle x^3y^2 \rangle$

$\langle x^3y \rangle = \{x^3y, xy^2, e\} = \langle xy^2 \rangle$

$\langle x^4y \rangle = \langle x^6y^2 \rangle$

$\langle x^5y \rangle = \langle x^4y^2 \rangle$

$\langle x^6y \rangle = \langle x^2y^2 \rangle$

$12 + 6 + 2 + 1 = 21$

$$(xy)(xy) = x(yx)y = xx^4yy = x^5y^2$$

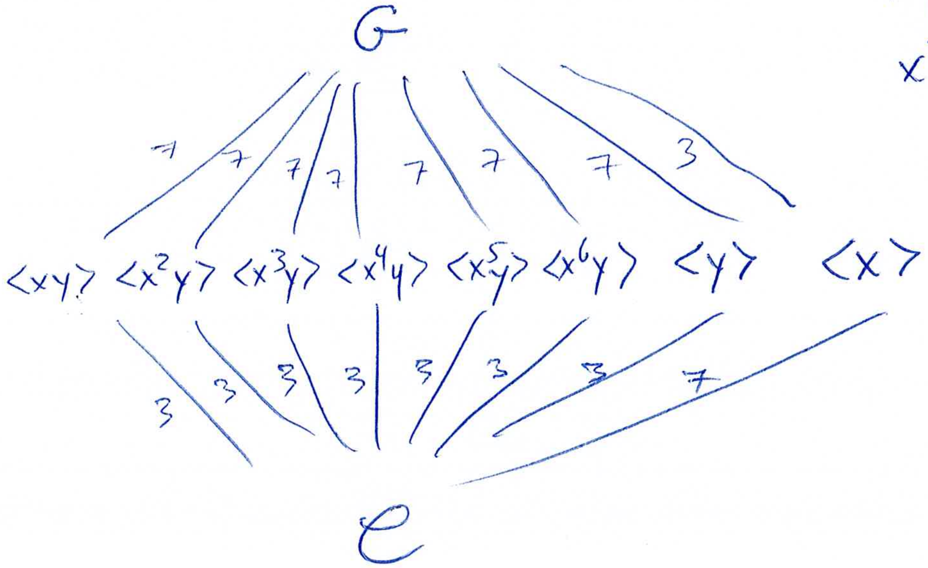
$$(xy)^3 = xy(xy)^2 = xyx^5y^2 = xx^6yy^2 = e$$

$$x^2yx^2y = x^2xyy = x^3y^2$$

$$(x^2y)^3 = (x^2y)(x^3y^2) = x^2x^5yy^2 = e$$

$$x^3y\cancel{x}^3y = x^3x^5yy = xy^2$$

$$x^3yxy^2 = x^3x^4yy^2 = e$$



conj

$$x(xy)x^{-1} = x^2yx^{-1} = x^2yx^6 = x^2x^3y = x^5y$$

(5)

$G = \langle x, y \mid x^8 = y^2 = e \quad xy = yx^2 \rangle$

$x^2 y = x\,xy = xyx^2$
$= yx^2x^2 = yx^4$

$x^3 y = xyx^4 = yx^6$

$x^4 y = x\,x^3y = xyx^6 = \underline{yx^8 = y}$

$x^4 y = y \Rightarrow \underline{x^4 = e}$

$G \cong \mathbb{Z}_2$

$x^2 y = yx^4 = y \Rightarrow x^2 = e.$

$xy = yx^2 = y \Rightarrow \underline{\underline{x = e}}$

$x^8 = y^2 = e \quad xy = yx^3$

$x^2 y = x\,xy = \cancel{x}yx^3$
$= yx^6$

$x^3 y = xyx^6 = yx^9 = yx$

$x^4 y = xyx = yx^3x = yx^4$

$x^5 y$

$x^6 = y^3 = e, \; xy = yx^3$

$x^2 y = \cancel{e}\, xyx^3 = yx^3x^3 = y \Rightarrow x^2 = e.$

$\underline{xy = yx^3 = yx}$ so $\underline{\mathbb{Z}_2 \times \mathbb{Z}_3}$

$\forall g \in G$ if $H \leq G$, then $gHg^{-1} \leq G$.

$\underline{\text{Prop}}$ $|H| = |gHg^{-1}|$ $\quad$ $\underline{\text{Define}}$ φ