2/26

Goal for today: dihedral groups, products,

$GL_2(\mathbb{Z}_p)$

3 classes + then spring break.

$\mathbb{Z}_2 = \{0,1\}$

$+$	0	1
0	0	1
1	1	0

$U(\mathbb{Z}) = \{1,-1\}$

$\cdot$	1	-1
1	1	-1
-1	-1	1

isomorphism

homomorphism

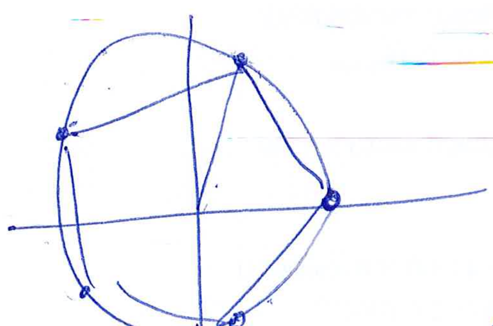
dihedral groups groups of symmetries.

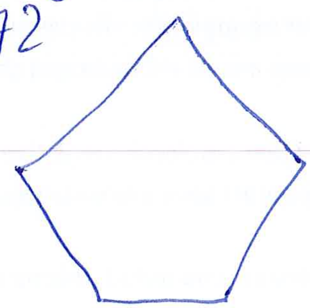
n-gon (regular)

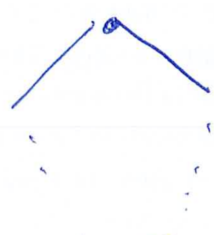
equlateral - all sides have same length

equangular all angles have same degrees

$\frac{360^\circ}{5} = 72^\circ$







2

group of rigid motions of the n-gon

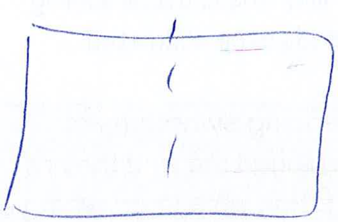
Dihedral group D_n

R_{90° - rotation by 90° - clockwise.

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $r = (1234)$ $r^2 = (13)(24)$

$r^{-1}(1432) = r^3$

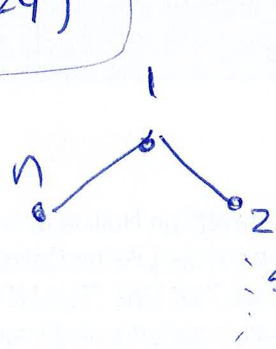
$F_{y\text{-axis}}$



$s = (12)(34)$

$D_4 = \{e, r, r^2, r^3, s$

$\boxed{(12)(34),\ (14)(23),\ (13),\ (24)}$



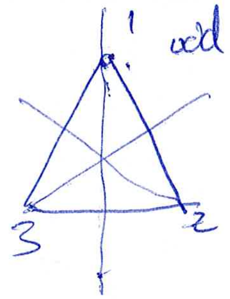
n-choice

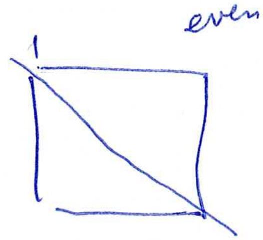
we can send 1 to n choices, say i and send 2 to either i+1, i-1

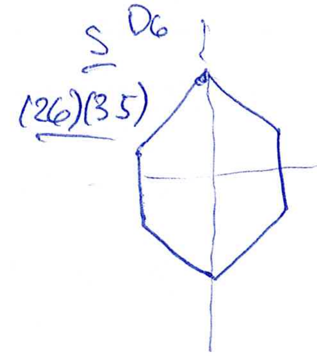
$|D_n| = 2n$

So there are exactly $\underline{2 \cdot n}$ possibilities

3







$\{e, (12), (13), (23), (123), (132)\} = S_3 = D_3$

$D_4 \neq S_4$

$D_n \quad r = (123\ldots n)$

s is a flip not a rotation

$D_n \quad r^n = e \qquad s^2 = e \qquad o(s) = 2$

$0 \le i < j < n$

$= \{r^i\} = \left\{\begin{matrix} e, r, r^2, \ldots, r^{n-1} \\ s, sr, sr^2, \ldots, sr^{n-1} \end{matrix}\right\}$

2n-many elements

$sr^i = sr^k$

$sr^i = r^j$

$s = r^{i-j}$ ✗.

$rs(1) = 2 \qquad sr(1) = 4$

$rs(2) = 1$

D_n is not abelian $\forall n \ge 3$

D_n is generated by 2-elements r, s

$D_2 \cong \mathbb{Z}_2 \cong U(8)$

not cyclic

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1-gen   1 variable g. $g^n = e$ or $o(g) = \infty$

$o(g) = n$

$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$

(finite)   2-gen x, y $x^n = 1$, $y^m = 1$

$o(x) = n$ $o(y) = m$

x^k, y^k $x^k y^j$ yx $xy =$

$\underbrace{xyxyxyxy}_{(xy)^4}$, $\underbrace{xxxxyyyxxx}_{x^4 y^3 x^3}$

$xy = y^k x^j$

$o(x)=n,\ o(y)=m$

finite 2-gen $\langle x, y \mid x^n = e, y^m = e \quad xy = y^k x^j \rangle$

5

$$\boxed{D_n = \langle r, s \mid r^n = e = s^2,\ rs = sr^{-1} \rangle}$$

$(rs = sr^{n-1}$

D_5 $\boxed{r^3 s r^2 s r = r^2}$

$rs = sr^4$, $sr = rs^{k}$

$\cancel{sr^4 = rs}$

$sr \cdot r^3 = r^k s r^3$

$s = r^k s r^3$

$r^{-k} s =$

$rs = sr^4$

$rsr = sr^5 = s$

$sr = r^{-1} r s r = r^{-1} s = r^4 s$

$r^3 s r^2 s r = s^i r^k$

$r^2 s = s r^{-1}$

$r(rs) = rsr^{-1}$

$= sr^{-1} r^{-1} = sr^{-2} = \underline{sr^3}$

$r^3 s = rsr^3 = sr^{-1} = sr^2$

$r^3 s r^2 s r$

$sr^2 s r^3 r$

$sr^2 s r^4$

$\cancel{s} s r^3 r^4 = r$

6 2-ger $G = \langle x, y \mid x^n = e\ y^m \quad xy = y^j x^k \rangle$

$e, x, \dots, x^{n-1}$

$y, yx, \dots, yx^{n-1}$

$y^2, y^2 x, \dots, y^2 x^{n-1}$

$\vdots$

$y^{m-1}, y^{m-1} x, \dots, y^{m-1} x^{n-1}$

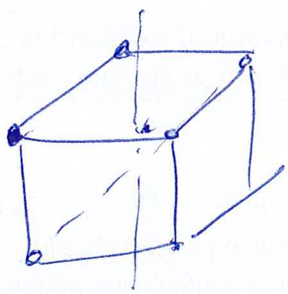
6 $D_n = \langle r, s \mid r^n = e = s^2,\ rs = sr^{-1} \}$

$G = \langle x, y \mid \ldots \rangle$

$o(xy)$

$\underline{xyxy}$

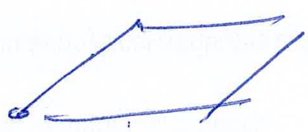
$(xy)(\quad) \ldots$

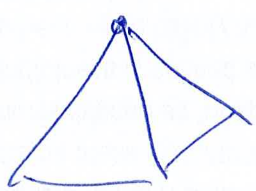


$8 \cdot 3 = 24$

$6 \cdot 4 = 24$

$\underline{S_4}$





7

$GL_2(\mathbb{R})$

$SL_2(\mathbb{R})$

$GL_2(\mathbb{Z}_p)$ p field.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

R commut ring $\quad M_2(R) \quad U(M_2(R))$

$\det(R) \neq 0$

$|GL_2(\mathbb{Z}_p)|$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|\mathbb{Z}_p \times \mathbb{Z}_p| = p^2$

$\boxed{(p^2-1)(p^2-p)}$

$GL_2(\mathbb{Z}_2) \cong S_3$

$\mathbb{Z}_3 \quad \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

$(3^2-1)(3^2-3)$

$\frac{8 \cdot 6}{48}$

Extra Credit

if you find $\langle \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \rangle_{GL_2(\mathbb{Z}_3)} \quad o\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} =$

1,