2/17

Quiz: multiplying elements in S_n.

- Exam: next Tuesday in class.
  - posted some old exams 1 & a review
    Canvas > Files > Review

Ex Quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \quad \text{so } |Q_8| = 8.$$

mult. resembles cross product on $\mathbb{R}^3$

$i \cdot j = k \qquad ji = -k \qquad$ 1 is the identity

$jk = i \qquad kj = -i \qquad (-1)^2 = 1$

$ki = j \qquad ik = -j$

$i^2 = j^2 = k^2 = -1$

$(i)(-i) = -i^2 = -(-1) = 1$

Q_8 not abelian

$\langle i \rangle = \{i, -1, -i, 1\} = \langle -i \rangle$

$\langle j \rangle = \{j, -1, -j, 1\} = \langle -j \rangle$

$\langle k \rangle = \{k, -1, -k, 1\} = \langle -k \rangle$

$\langle -1 \rangle = \{-1, 1\}$

~~4=o(i)~~

$4 = o(\pm i) = o(\pm j) = o(\pm k)$

$2 = o(-1)$

$1 = o(1)$

Lattice of subgroups

$$\begin{tikzcd}[column sep=small]
 & Q_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle i \rangle \arrow[dr, dash] & \langle j \rangle \arrow[d, dash] & \langle k \rangle \arrow[dl, dash] \\
 & \langle -1 \rangle \arrow[d, dash] & \\
 & \langle 1 \rangle &
\end{tikzcd}$$

every proper subgroup is cyclic (this is true for S_3).

Exs groups

$(\mathbb{Z}_n, +)$ $\quad (\mathbb{Z},+), (\mathbb{Q},+), (\mathbb{R},+)\ (\mathbb{C},+)$.

$(\mathbb{Q}^*, \cdot), (\mathbb{R}^*, \cdot)\ (\mathbb{C}^*, \cdot)$

$U(R)$

$GL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid AX = I_2 \text{ has a soln}\}$
$= \{ \qquad \mid \det(A) \neq 0\}$

$SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) \mid \det(A) = 1\}$

$U(\mathbb{Z}_n) \qquad |U(\mathbb{Z}_n)| = \phi(n)$

Q_8, S_n

---

in S_n What is a transposition? 2-cycle $(a\,b)$

the order of a transposition is $\underline{2}$

$(\text{i.e. } \sigma = \sigma^{-1})$ if $|\sigma| = 2$ must σ be a transposition. a,b,c,d all distinct.

no! $(ab)(cd)$

~~Def~~ Any K-cycle can be written as a (nonunique) product of transpositions (not disjoint).

$(123) = (13)(12)$ $\qquad$ $(231) = (21)(23)$

$(375) = (35)(37)$

$(2376) = (26)(27)(23)$

the K-cycle can be written as the product of K-1 transposition

$(123) = (13)(12)\ (34)(34)\ (13)(13)$

$\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ and say σ_i, σ_j are disjoint

$\sigma_i \Rightarrow k_i$-cycle.

$\sigma =$ (transp)(trans)( ) ... (transp).

is a product of transpositions.

<u>Prop</u> Every permutation can be written as a product of transpositions.

<u>Def.</u> σ is an even permutation if it can be written as an even-numbered product of transpositions.

when is a K-cycle an even perm. when K is odd

(4) <u>Fact</u>: a permutation cannot be both even + odd.

σ <u>odd.</u> $\sigma = (a_1 b_1)(a_2 b_2)\ldots(a_{2k+1} b_{2k+1})$ $a_i \neq b_i$

σ $e = \ldots\ldots$

$e = (12)(12)$ <u>e is even</u>

the trick is to show <u>e is not odd.</u>

$$A_n = \{\text{even permutations}\}$$

$e \in A_n$

<u>σ is even.</u> $\cancel{\sigma^{-1} = (a}$

$\cancel{(a_1 b_1) \ldots (a_{2n} b_{2n}}$ $\sigma = (a_1 b_1) \ldots (a_j b_j)$

$\sigma^{-1} = (a_j b_j)(a_{j-1} b_{j-1}) \ldots (a_1 b_1)$

σ, τ even $\sigma \cdot \tau$ is even

<u>Prop</u> $A_n \leq S_n$ $|A_n| = \dfrac{n!}{2}$

⑧ $|A_4| = 12$

no transp is even

(12)

$$\boxed{A_4 = \{ e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134)(143), (234)(243) \}}$$

$2, 2$-cycles + 3 cycles.

---

if σ + τ are disjoint then

$$\sigma \circ \tau = \tau \circ \sigma$$

Pf: Let $i \in \{1, \dots, n\}$

Can i appear in the cycle decomp of both σ, τ no b/c they are disjoint.

3 cases a) i is in cycle decom of σ

b) i " " " " τ

iii) i is in neither

Case iii) $(\sigma \circ \tau)(i) = \sigma(\tau(i)) = \sigma(i)$
$= i$

$\tau(i) = i$

$\sigma(i) = i$

$(\tau \circ \sigma)(i) = \tau(\sigma(i)) = \tau(i) = i.$

Case 1) $\sigma(i) = j$ $\quad j \neq i$ $\quad \tau(i) = i$

$\sigma = (\quad)(i j)(\quad)$ $\quad$ j in cycle decom of σ so not in τ

$\tau(j) = j$

$(\sigma \circ \tau)(i) = \sigma(\tau(i)) = \sigma(i) = j$

$(\tau \circ \sigma)(i) = \tau(\sigma(i)) = \tau(j) = j.$

so $(\sigma \circ \tau)(i) = (\tau \circ \sigma)(i)$ $\forall i$

so equal.

⑥ In S_n disjoint cycles commute.

how many 2-cycles in S_n. $\binom{n}{2} = nC_2 = C^n_2$

3-cycles $\binom{n}{3} \cdot 2$

# of perm of k many things is $k!$

$(a\,b\,c) = \cancel{(b\,c\,a)}$

$(a\,b\,c\,d)$

4-cycles $\binom{n}{4} \cdot 3!$

k-cycles $\binom{n}{k}(k-1)!$

5-cycles $\binom{n}{5} \cdot 4!$

2,2-cycles $\dfrac{\binom{n}{2}\binom{n-2}{2}}{2}$

$(ab)(cd) = (cd)(ab)$

ord

2,2,2 cycles $\dfrac{\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}}{3!}$

if σ k-cycle, τ j-cycle. σ, τ disjont

$o(\sigma) = k$ $\quad o(\tau) = j$ $\quad o(\sigma \cdot \tau) = \operatorname{lcm}(k, j)$

⑦ take $\sigma \in S_n$.

$\sigma = \sigma_1 \cdots \sigma_n$ $\quad$ σ_i ∋ k_i-cycle

$o(\sigma) = \operatorname{lcm}(k_1, \dots, k_n)$

$n = 3, \dots, 10$

find elements of largest possible order.

$n=3$ $\quad$ $\underline{3}$

$\cancel{n}=5$ $\quad$ 5 $\quad$ 2,3 - $\underline{6}$

$n=6$

$n=9$ $\quad$ 5,4 = 20

$n=10$ $\quad$ 7,3 21

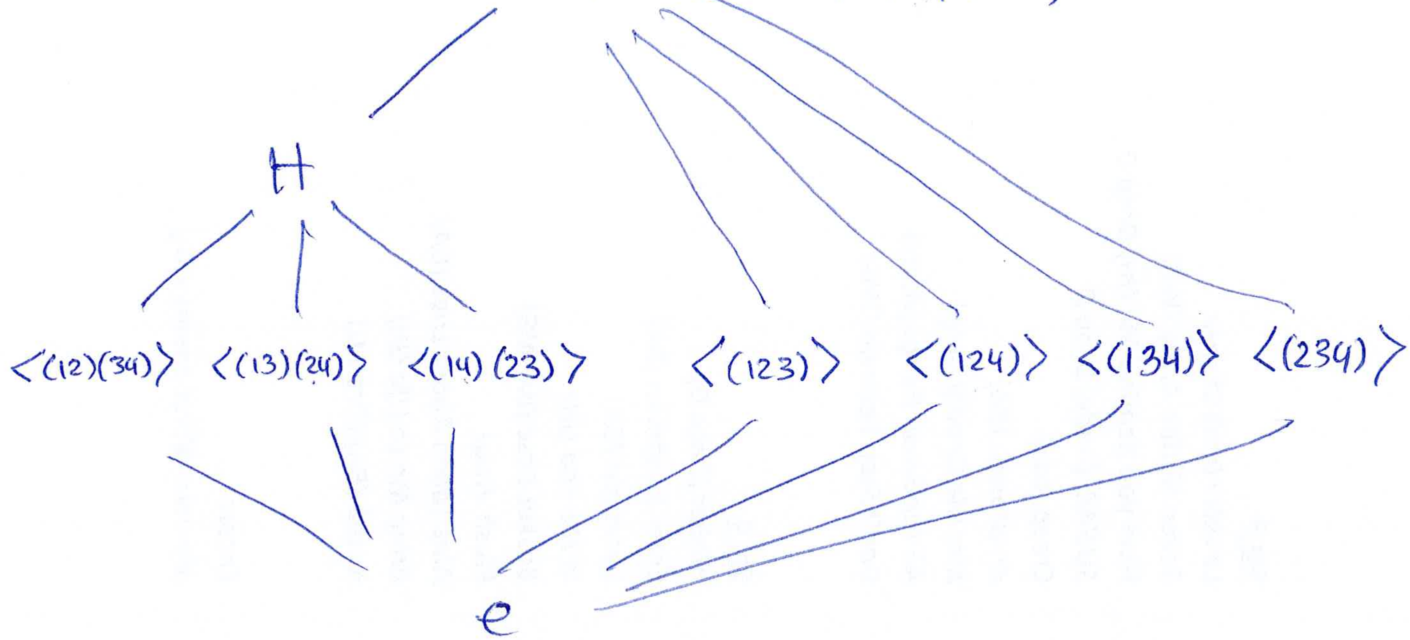
2,3,5 = 30

---

lattice of A_4

(8) A_4 $|A_4| = 12$ possible sizes of subgroups of A_4

1, 2, 3, 4, 6, 12



V is a subgroup either V is cyclic or it has 3 elements of

$(12)(34)(13)(24) = (14)(23)$

$(123)(124) = (13)(24)$

---

differentrate $o(\sigma)$ + the parity of σ even or odd.

✓ if $o(\sigma)$ is odd, then σ is even.

if $o(\sigma)$ is even, then σ can be even or odd

8

$(12)(34)\cdot(123) = (243)$

$(123)(12)(34) = \cancel{(13)}(134)$

$\langle (12)(34), (123) \rangle = A_4$

$(13)(24)(123) = (142)$

$(123)(13)(24) = (243)$

Cont.

no subgp of order 6.