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- construct some lattice of subgps. of a given group.

- $\langle g \rangle$ each of these.

Lemma if $h \in \langle g \rangle$, then $\langle h \rangle \leq \langle g \rangle$.

$$\begin{array}{c} G \\ \subseteq \\ \langle g \rangle \\ | \\ \langle h \rangle \\ \langle e \rangle \end{array}$$

Pf: $\exists\, n \in \mathbb{Z}$ $h = g^n$.

Take $x \in \langle h \rangle$, $x = h^m$ some $m \in \mathbb{Z}$.

$= (g^n)^m = g^{mn} \in \langle g \rangle$.

Def the center of the group G is the set

$$Z(G) = \{ g \in G \mid \forall x \in G,\ gx = xg \}.$$

the thing that commute w/ everything in G.

Ex G abelian iff. $Z(G) = G$.

In general, $e \in Z(G)$.

$\cancel{ex = gx} = g$

$ex = xe = x\ \forall x$.

(HW) Prop $Z(G) \leq G$.

Pf: $e \in Z(G)$, $g_1, g_2 \in Z(G)$

g

1) $g_1 g_2 \in Z(G)$

2) $\cancel{g \in Z(G)}$.

$g^{-1} \in Z(G)$

~~$\mathcal{C}$~~ **Def.** $g \in G$.

$$C_G(g) = \{x \in G \mid gx = xg\}.$$

"the centralizer of g"

~~$e \in G$~~

$e \in C_G(g)$
$g \in C_G(g)$
$g^{-1} \in C_G(g)$

$\langle g \rangle \leq C_G(g)$

Prop $\forall g \in G$, $C_G(g) \leq G$.

$Z(G) \subseteq C_G(g)$

$Z(G) = \bigcap_{g \in G} C_G(g)$

Pf: $e \in C_G(g)$ since $eg = ge = g$.

Let $g_1, g_2 \in C_G(g)$ wts $g_1 g_2 \in C_G(g)$

$$(g_1 g_2) g \overset{\text{assoc}}{=} g_1(g_2 g) \overset{g_2 \in C_G(g)}{=} g_1(g g_2) \overset{\text{assoc}}{=} (g_1 g) g_2 \overset{g_1 \in C_G(g)}{=} (g g_1) g_2 \overset{\text{asso.}}{=} g(g_1 g_2).$$

$(ab)^{-1} = b^{-1}a^{-1}$

~~$g \in G$ wts~~ $g_1 \in C_G(g)$ wts $g^{-1} \in C_G(g)$.

means $g_1 g = g g_1$, take inverses of both sides

$$g^{-1} g_1^{-1} = (g_1 g)^{-1} = (g g_1)^{-1} = g_1^{-1} g^{-1}$$

mult. on left by g.

$$g_1^{-1} = g g_1^{-1} g^{-1}$$

mult on right by g

$$g_1^{-1} g = g g_1^{-1}. \quad \square$$

so $g_1^{-1} \in C_G(g)$.

③

Fact $gx = xg$ iff $gxg^{-1} = x$

iff $g^{-1}xg = x$

Def. ~~the~~ an element of the form

gxg^{-1} is called a <u>conjugate of x</u> (for some g).

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h is a conjugate of x if $\exists\, g \in G$ s.t. $h = gxg^{-1}$

conjugate $h = g^{-1}xg$.

$x^g = g^{-1}xg$

Prop a collection of subgroups indexed by I

$\{H_i\}_{i \in I}$ indexing them
I is index set.

$\{H_i\}_{i=1}^{\infty} = \{H_i\}_{i \in \mathbb{N}}$

$= \{H_1, H_2, H_3, \dots\}$

Given $\{H_i\}_{i \in I}$, $\bigcap_{i \in I} H_i \leq G$.

arbitrary of intersection of subgroups is again a subgroup.

4 $\{H_i\}_{i\in I}$ subgps

those elements that are in every H_i

$$\bigcap_{i\in I} H_i = \{x\in G \mid \forall i\in I,\ x\in H_i\}$$

Pf. $\underline{e\in\bigcap H_i}$: let $i\in I$, $H_i\le G$ $\therefore e\in H_i$

Next let $g\in\bigcap H_i$ WTS $g^{-1}\in\bigcap H_i$

Let $i\in I$. then $g\in H_i$ (def. of int.)

~~but~~ then $g^{-1}\in H_i$ b/c $H_i\le G$.

Let $g_1,g_2\in\bigcap H_i$. Let $i\in I$ $g_1,g_2\in H_i$

$g_1g_2\in H_i$ $H_i\le G$.

$g_1g_2\in\underline{\bigcap H_i}$ ∎

Let $S\subseteq G$.

Take set $W=\{K\le G \mid S\subseteq K\}$. $G\in W$.

subgp gen. by S. Let $\langle S\rangle = \underline{\bigcap W}$ the intersection of all subgps containing S.

$S\subseteq\langle S\rangle\in W$.

$S=\{g\}$ $\langle S\rangle=\langle\{g\}\rangle=\langle g\rangle$

(5) $\underline{H, K \leq G}$

$$\begin{tikzcd}[column sep=small]
 & G & \\
 & \langle H,K \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & H \cap K &
\end{tikzcd}$$

lattice

Lagrange's Theorem if $H \leq G$. then $|H| \,\big|\, |G|$

Pf. ~~Left~~ left cosets $\overset{H \leq G}{}$ $g \in G$

left coset w/ rep. g. $\quad gH = \{x \in G \mid \exists h \in H \;\; x = gh\} = S_1$

$$= \{x \in G \mid x^{-1} g \in H\} = S_2$$

take $x \in S_1$ $\exists h \in H$ $x = gh$. $\quad x^{-1} = h^{-1} g^{-1}$

$$x^{-1} \cdot g = h^{-1} g^{-1} g = h^{-1} \in H. \quad \text{so } x \in S_2. \qquad S_1 \subseteq S_2.$$

---

take $x \in S_2$ $\quad x^{-1} g \in H. \quad \Rightarrow g^{-1} x \in H$

$$x = g(\underbrace{g^{-1}x}_{\in H}) \in gH = S_1 \quad \text{so } \underline{S_2 \subseteq S_1}$$

Define a relation on G by $g \sim x$ iff $x^{-1} g \in H$.

WTS $\sim$ is an equivalence relation.

(6) $g \sim g$? $g^{-1} \cdot g \in H$? yes $e \in H$.

if $g \sim h$, then $h \sim g$ $\quad g \sim h \Rightarrow h^{-1}g \in H. \Rightarrow g^{-1}h \in H$

$\Rightarrow h \sim g$

if $g \sim h$, $h \sim K$, then $g \sim K$

$h^{-1}g \in H.$ $\quad K^{-1}h \in H.$ $\quad$ so $(K^{-1}h)(h^{-1}g) \in H$

$= K^{-1}g \in H$

So $g \sim K$.

So $\sim$ is an equivalence relation.

so the equivalence classes partition G

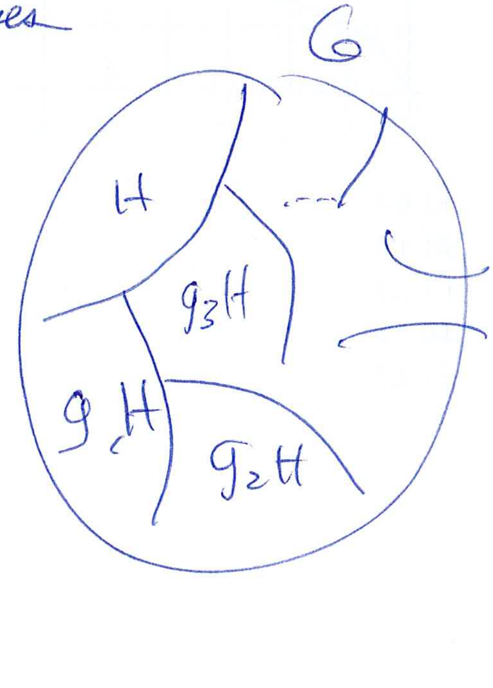
$[g] = \{x \in G \mid g \sim x\} = \{x \in G \mid x^{-1}g \in H\}$

$= gH.$

So the cosets are the equiv. classes

and they partition G.

$[e] = eH = H$



⑦. take an arbitrary coset gH.

define $\varphi_g : H \to gH$

$\varphi_g(h) = gh$

if $h_1 = h_2$ well-defined

$\varphi_g(h_1)\ gh_1 = gh_2 = \varphi_g(h_2)$

if $\varphi_g(h_1) = \varphi_g(h_2)$ $\quad gh_1 = gh_2$ mult left g^{-1}

$h_1 = h_2$

So φ_g is injective. (1-1).

Let $x \in gH$. $\exists h \in H$ s.t. $x = gh$.

$\varphi_g(h) = gh = x$ $\quad \varphi_g$ is surjective (onto)

So φ_g is a bijection

$|H| = |gH|$. $\quad |G| = \sum |gH|$

$= \sum |H| = (\text{numb. of cosets of } H) \cdot |H|$

$[G:H] = \frac{|G|}{|H|}$.

$|H| \mid |G|$

the index of H.

$[G:H]$

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$\cancel{\mathbb{Z}}$ $[\mathbb{Z} : n\mathbb{Z}] = n.$

$a \equiv b \bmod$ if $b - a = nk \quad k \in \mathbb{Z}.$

$b - a \in n\mathbb{Z}$

$-b + a \in n\mathbb{Z}$