1/22

- last quiz on operations. Please let me know if you are still struggling on this topic.
- moving forward we will assume that we are a group.

"f be a function from A to B"

Review Let $f: A \to B$.

Def 8. a) f is said to be <u>injective</u> (or one-to-one, or 1-1)

if $\forall a_1, a_2 \in A$, $(f(a_1) = f(a_2) \to a_1 = a_2)$.

b) f is said to be <u>surjective</u> (or onto)

if $\forall b \in B \; \exists a \in A \;\; f(a) = b$.

(f is inj. iff ~~if f(a₁)≠f(a₂), then a~~

if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$ "horizontal line test")

f surj iff $B = \text{range}(f)$)

c) if f is injective & surjective, then f is called <u>bijective</u>

(or a "one-to-one correspondence".)

Def. 9 A bijection from A onto itself is called

a <u>permutation</u>.

(2) Let A be a set. A^A $\mathcal{F}(A,B) = B^A$

We denote $\underline{\mathcal{F}(A,A)}$ the set of all function from A to A.

We define an operation on $\tilde{\mathcal{F}}(A,A)$ (called composition) as for $f,g \in \tilde{\mathcal{F}}(A,A)$ $f \circ g$ is the function defined by: for each $a \in A$ (A as the domain)

$$(f \circ g)(a) = f(g(a)).$$

this makes sense as $a \in A$ $\Rightarrow g(a) \in A$ and so $f(g(a)) \in A$.

$f(x) = x^2$
$g(x) = \sin x$
$(f \circ g)(x) = f(\sin x) = (\sin x)^2 = \sin^2 x$

I. $\circ$ is associative: Let $f,g,h \in \mathcal{F}(A,A)$

We show that $(f \circ g) \circ h = f \circ (g \circ h)$.

Recall to show that two functions are equal we show that they agree for all $a \in A$.

$$((f \circ g) \circ h)(a) \overset{\text{def. of comp. applied to } f\circ g, h}{=} (f \circ g)(h(a)) = f(g(h(a)))$$

def. of com. applied to f,g.

$$(f \circ (g \circ h))(a) \overset{\text{def comp. app to } f, (g\circ h)}{=} f((g \circ h)(a)) = f(g(h(a)))$$

def. of comp. applied to g,h.

So $\circ$ is associative.

II. ~~$\circ$ has~~ $\exists$ a $\circ$-identity. Define $i : A \to A$ by $i(a) = a$.

Then $i \in \tilde{\mathcal{F}}(A,A)$. We let $f \in \tilde{\mathcal{F}}(A,A)$ be arbitrary and show $f \circ i = f$

$(f \circ i)(a) = f(i(a)) = f(a)$. ✓ $\underline{(f \circ i)(a)} = f(a)$ $\forall a \in A$ so

$i \circ f = f$ $\quad (i \circ f)(a) = i(f(a)) = f(a)$ ✓.

③ Proposition $\boxed{(\mathcal{F}(A,A), \circ, i)}$ $\mathcal{F}(A,A)$ is a monoid.

$\forall x \in A$ $f(x) = a$.

Ex. Let $A = \{a,b\}$. $a \neq b$ Define $f = \bar{a}$ ✓ constant function (so $f(a) = f(b) = a$.) $\in \mathcal{F}(A,A)$.

We claim that $\nexists g \in \mathcal{F}(A,A)$ s.t. $f \circ g = i$.

Pf. by contradiction: Suppose $g \in \mathcal{F}(A,A)$ and $f \circ g = i$.

Then $(f \circ g)(b) = i(b) = b$.

But $f(g(b)) = a$ and $a \neq b$.

This works for any set A as long as $|A| > 1$.

So in general $\mathcal{F}(A,A)$ is not a group. (iff $|A| = 1$)

Def 10 Take the set of all permutations on A.

We denote this set by ~~Perm(A)~~

$(S(A), \circ)$ $\circ \equiv$ composition

zero ↓ O $\circ$ comp

f be a perm. on A. Since f is inj it has an inverse function f^{-1}

the $\mathrm{dom}(f^{-1}) = \mathrm{range}(f) = A$ f onto

f^{-1} is a permutation

$(f \circ f^{-1})(x) = x$

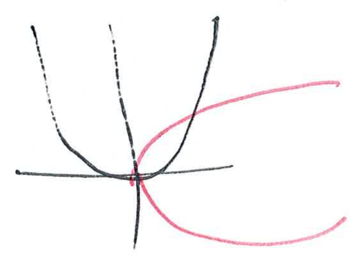
$= f(f^{-1}(x)) = x$

$= i(x)$

$\forall x \in A$

$f^{-1} \circ f(x) = x$

$= i(x)$



Thm $(S(A), \circ, i)$ is a group.

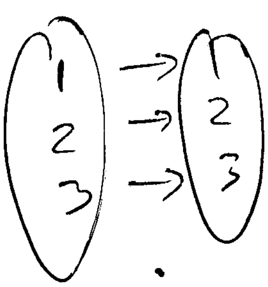
Notation when $\underline{A=\{1,\dots,n\}}$ instead of $S(A)$ we write $\underline{S_n}$

called the symmetric group on $\{1,\dots,n\}$

$(S_n, \circ, i)$ is a group

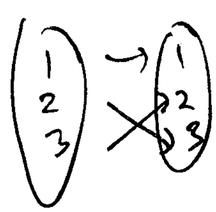
Question $|S_n| = \underline{n!}$

$\underline{S_3}$



i

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



<u>cycle notation</u>

$(1)(23) \to ($

(23)

(123)

(5) S_5

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$$

$f \in S_5$

$f(3) = 4$

$f(2) = 2$

$f(4) = 5$

$f(1) = 3$

$f(5) = 1$

$$(1345)(2) = \underline{(1345)}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$(132)(45) = (45)(132)$

3,2-cycle

2,3-cycle

S_5

$\sigma = (1234)$

4-cycle.

$\tau = (14)(25)$

2,2 cycle.

$(1234) = (2341) = (3412)$

$= (4123)$

$\neq \underline{(1243)}$

6

$\sigma = (1234)$, $\tau = (14)(25)$

right to left

$\sigma \circ \tau = (1234)(14)(25)$

$1 \to 4 \to 1 \qquad 1 \to 1$

$(1)(2534)$

$\sigma \circ \tau = (2534)$

$\tau \circ \sigma = (14)(25)(1234)$

$\tau \circ \sigma = (1523)$

$\sigma \circ \tau \neq \tau \circ \sigma$

S_5 is not abelian

<u>Cor</u> S_n is abelian iff $n \leq 2$

$(123)(12) = (13)$

$(12)(123) = (23)$

not abelian

$(S_3, \circ, i)$

$|S_3| = 6$.

MAS 4301 – Functions

In class we discussed the collection of functions from a fixed set S into the real numbers. The appropriate set-theoretic notation would be $\mathbb{R}^S$ but I have found that this notation is not so suggestive. So let's denote this set by $\mathcal{F}(S, \mathbb{R})$. Here the $\mathcal{F}$ can remind you of functions, and then S goes first since it is the domain, and then $\mathbb{R}$ since it is the co-domain.

In class we defined an operation on $\mathcal{F}(S, \mathbb{R})$ as follows: for $f, g \in \mathcal{F}(S, \mathbb{R})$, we defined $f \oplus g$ to be the function that sends each $s \in S$ to the real number $f(s) + g(s)$. We showed that the operation is associative and commutative. We also showed that there is a $\oplus$-identity: the function $\bar{0}$ which sends everything in S to $0 \in \mathbb{R}$.

$$\bar{0} \oplus f = f.$$

We also showed that each function $f \in \mathcal{F}(S, \mathbb{R})$ has an $\oplus$-inverse. It is the function that sends each $s \mapsto -f(s)$. Suggestively, we denote this function by $-f$. It follows that

$$f \oplus -f = \bar{0}.$$

Remark 0.1. The big idea in dealing with functions is that to show two functions $f, g \in \mathcal{F}(S, \mathbb{R})$ are equal, you have to show that **for all** $s \in S$, $f(s) = g(s)$.

Example 0.2. Let $T = \{0, 1\}$ and let $f, g, h \in \mathcal{F}(T, \mathbb{R})$ be defined by $f(t) = 2t+1$, $g(t) = 1+4t-2t^2$, and $h(t) = 5^t + 1$.

i) Show that $f = g$.

$f(0) = 2(0)+1 = 1$ | $f(1) = 2(1)+1 = 3$ | $(f \oplus g)(x) = f(x) + g(x)$

$g(0) = 1+4(0) - 2 \cdot 0^2 = 1$ | $g(1) = 1+4(1) - 2 \cdot 1^2 = 3$.

so f, g agree on all pts. in T. $\therefore f = g$

ii) Show that $f \oplus g = h$.

$(f \oplus g)(0) = f(0) + g(0) = 1+1 = 2$ | $(f \oplus g)(1) = 3+3 = 6$

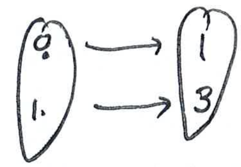
$h(0) = 5^0 + 1 = 1+1 = 2$ | $h(1) = 5^1 + 1 = 6$

$\therefore f \oplus g = h$

iii) Is f injective?

$f(0) = 1$

$f(1) = 3$



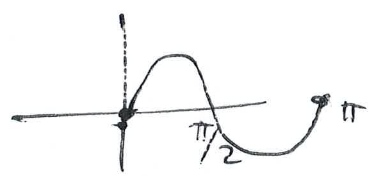
injective.

iv) Let $j \in \mathcal{F}(T, \mathbb{R})$ be defined by $j(t) = \sin(\pi t)$. True or False: $j = \bar{0}$.

$j(0) = 0$

$j(1) = 0$

so $j = \bar{0}$



v) Is f surjective? If no, is there any $k \in \mathcal{F}(T, \mathbb{R})$ that is surjective.

no & no b/c T only has 2 pts. + $|k(T)| \leq 2$
$\leq |\mathbb{R}|$

vi) If we take $T' = \{0, 1, 2\}$ are i), ii), iii) and iv) still true?