1/20

- Daily Plan in Canvas : switch to assignments by type.

- Quiz 2

- HW #1 is up on Canvas : Due Jan 29^{th}

Def 5. 1) A set A equipped w/ a binary operation that is associative is called a semi-group

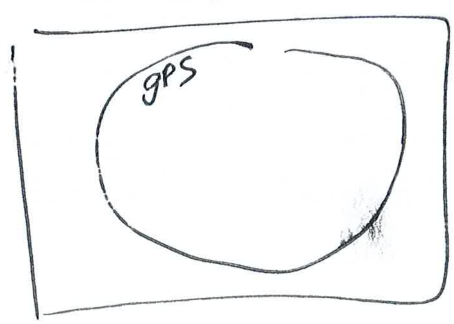
" $(A,*)$ is a semi-group.

2) if $(A,*)$ is a semi-group and there is a $*$-identity, then we say $(A,*,e)$ is a monoid.
(call it e)

Note I do hope to see improvement on the quizzes. So please work on this. I will drop a quiz (maybe 2).

A group is a monoid. A monoid is a semi-group.

at some pt



$(a^{-1})^{-1} = a$

Def 6 If $(A, *, e)$ is a group.

We call A abelian if $*$ is commutative.

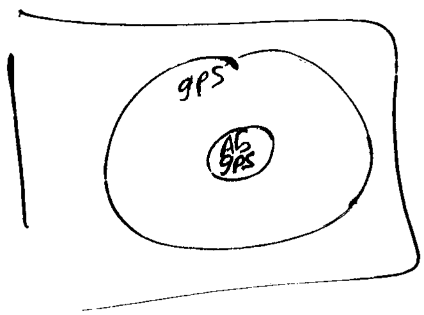
$a * b = a$

$b * a = b$

$A = \{1\}$

Ex. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ are all abelian

Abel



Ex $(a,b) * (c,d) = (a+2c, b+3d)$

$(c,d) * (a,b) = (c+2a, d+3b)$

$(1,2) * (3,4) = (7, 14)$

$(3,4) * (1,2) = (11, 10)$

not comm.

not assoc

Let's have some practice on operations.

Example 0.1. We define an operation on $A = \mathbb{R}^2$. First, recall that $\mathbb{R}^2$ is the set of ordered pairs of the form (a, b) for some $a, b \in \mathbb{R}$. A binary operation on R^2 must ake two ordered pairs and output a single ordered pair. Let $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$ and define

$$(a_1, a_2)\triangle(b_1, b_2) = (a_1 + a_2, b_1 b_2).$$

Should have been (a_1+b_1, a_2b_2)

(1) Compute $(1, 3)\triangle(2, -3)$. $= (4, -6)$

(2) We want to show this is commutative. We know what $(a_1, a_2)\triangle(b_1, b_2)$ equals so calculate

$$(b_1, b_2)\triangle(a_1, a_2)$$

$(2,-3) \triangle (1,3) = (-1,3)$ not commutative.

(3) Let's determine whether there is a $\triangle$-identity. We suppose that (e_1, e_2) is a $\triangle$-identity and then calculate two things and set them equal to each other. First, compute

$$(e_1, e_2)\triangle(a_1, a_2) = (e_1+e_2, a_1a_2)$$

and since it is an identity we set that equal to (a_1, a_2). So what is the $\triangle$-identity?

$= (a_1, a_2)$ no $\triangle$-identity

Example 0.2. On the set $A = \mathbb{R} \setminus \{0\}$ of non-zero real numbers define

$$r * s = r \cdot s^{\mathrm{sgn}(r)}.$$

Recall that $\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$

$2*(-1) = 2\cdot(-1)^1 = -2$

not comm.

(1) First, compute $(-1) * 2$. Then compute $2 * (-1)$. What does this tell you about the operation? $(-1)*2 = (-1)\cdot 2^{-1} = -\frac{1}{2}$

associativity

(2) Next, we are going to compute both $(r*s)*t$ and $r*(s*t)$ separately and then compare. We probably should prove this using cases.

Case 1: If $r, s > 0$, then $(r * s) * t = (rs) * t = rst$, while $r * (s * t) = r * (st) = rst$. Yes!!

Case 2: If $r > 0, s < 0$, then $(r * s) * t = (rs) * t = \frac{rs}{t}$. What is $r * (s * t)$? First use that $s * t = \frac{s}{t}$. $= r*(s/t) = r*(s/t) = \frac{rs}{t}$

BTW

$$r*s = \begin{cases} rs & r>0 \\ \frac{r}{s} & r<0 \end{cases}$$

Now you do the last two cases:

Case 3: $r<0, s>0$

$(r*s)*t = \frac{r}{s}*t = \frac{\frac{r}{s}}{t} = \frac{r}{st}$

$r*(s*t) = r*(st) = \frac{r}{st}$

Case 4: $r<0, s<0$

$(r*s)*t = (\frac{r}{s})*t = \frac{r}{s}\cdot t = \frac{rt}{s}$

$r*(s*t) = r*\frac{s}{t} = \frac{r}{s/t} = r\cdot\frac{t}{s} = \frac{rt}{s}$

(13)

Ex: Take an arbitrary set S $(S \neq \emptyset)$

$A = \{ f : S \to \mathbb{R} \}$ So an element of A is a function.

! how do we know when two functions are equal !

f, g are equal if $f(x) = g(x)$ $\forall x \in S$.

operation on fuctions

$$(f \oplus g)(x) = f(x) + g(x)$$

Pf $+$ is assoc

WTS $(f \oplus g) \oplus h = f \oplus (g \oplus h)$

Let $x \in S$. $((f \oplus g) \oplus h)(x) = (f \oplus g)(x) + h(x)$

$$= (f(x) + g(x)) + h(x)$$

assoc. of $+$ on $\mathbb{R}$

$\forall x$ $(f \oplus (g \oplus h))(x) = f(x) + (g \oplus h)(x)$

$$= f(x) + (g(x) + h(x))$$

$$(f \oplus g)(x) = f(x) + g(x) = g(x) + f(x) = (g \oplus f)(x)$$

(5)

$e \oplus f = f$ $\qquad$ $(e \oplus f)(x) = f(x) \ \forall x \in S$.

$(e \oplus f)(x) = e(x) + f(x) = f(x)$

$e(x) = 0$

So e is the constant function that sends everything in S to $0 \in \mathbb{R}$. call this function $\bar{0}$

$f \oplus g = \bar{0}$

$g(x) = -f(x)$ $\qquad$ $\underline{g = -f}$

$(\{f : S \to \mathbb{R}\}, \oplus, \bar{0})$ is ~~a~~ abelian group

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~~Aus~~

fix set S. $A = \{f : S \to S\}$

$f, g \in A$ $\quad$ $(f \circ g)(x) = f(g(x))$ $\qquad$ $\circ$ is associative.

$\exists$ a $\circ$-identity.

$e(x) = x \quad \forall x \in S$

not every function has a $\circ$-inverse

the ones are precisely the bijections.

<u>Def 7</u> a bijection from S to itself is called a permutation.