Name: ______________________________

Final Exam – MATH 3320

Directions: Make sure to show all necessary work to receive full credit. If you need extra space please use the back of the sheet with appropriate labeling.

(1) State the following definitions. $A$ is an $n \times n$-matrix and $\lambda \in \mathbb{F}$. (Make sure to include the correct quantifiers.)

(a) The scalar $\lambda$ is an *eigenvalue* of $A$ if

(b) The eigenspace corresponding to $\lambda$ is

$$E(\lambda) = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

(c) A function $T : V \to W$ between two vector spaces $V$ and $W$ is called a *linear transformation* if

(d) Given a linear transformation $T : V \to W$ the *nul space of* $T$ is the set

$\text{nul}(T) = \{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \}$

(e) The *characteristic polynomial* of $A$ is

(f) A set of vectors $\{v_1, v_2, \cdots, v_k\}$ is said to be linearly independent if

(g) The matrix $A = (a_{ij})$ is called *upper triangular* if

(h) The matrix $A$ is said to be *invertible* if

(i) The matrix $A$ is called *diagonalizable* if

(2) Let $S = \{ \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} \}$ be 3 vectors in $\mathbb{R}^3$. Is this set a linearly independent set? Explain.

(3) Determine whether the following system of equations is consistent or not. If it is consistent give the general solution. Give one specific example. Explain how you got your answer.

$$\begin{array}{ccccccc} x_1 & & & -2x_4 & = & -3 \\ & 2x_2 & +2x_3 & & = & 0 \\ & & x_3 & +3x_4 & = & 1 \\ -2x_1 & +3x_2 & +2x_3 & +x_4 & = & 5 \end{array}$$

(4) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 2 & 0 & 3 & 3 \\ 0 & -3 & 0 & 4 \end{bmatrix}$$

Use co-factor expansion to determine the determinant of $A$. Only use your calculator to check your answer.

(5) Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x, y) = (2x - 3y, 4x, x + y).$$

a) Prove that $T$ is a linear transformation.

b) Find $A_T$ the standard matrix representation of $S$. Then find $\text{rref}(A_T)$. Explain.

$A_T = \left[ \quad \right]$ and $\text{rref}(A_T) = \left[ \quad \right]$

c) Are the column vectors of $A_T$ linearly independent? Explain.

d) Do the column vectors of $A_T$ span $\mathbb{R}^3$? Explain.

e) Is $T$ one-to-one?

f) Is $T$ onto?

(6) Consider the following matrix over $\mathbb{R}$.

$$\mathbf{A} = \begin{bmatrix} 4 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) Find the characteristic polynomial for $\mathbf{A}$. b) What are the eigenvalues of $\mathbf{A}$ and what are their multiplicities? [Hint: $\lambda = 2$.] c) Determine whether $\mathbf{A}$ is diagonalizable? Explain your answer. If it is diagonalizable find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P^{-1}AP} = \mathbf{D}$.

(7) Consider the following matrix over $\mathbb{Z}_7$.

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$$

a) Find the characteristic polynomial for $\mathbf{A}$. b) What are the eigenvalues of $\mathbf{A}$ and what are their multiplicities? c) Determine whether $\mathbf{A}$ is diagonalizable? Explain your answer. If it is diagonalizable find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P^{-1}AP} = \mathbf{D}$.

(8) Let $T : V \to W$ be a linear transformation. Prove that the nul space of $T$ is a subspace of $V$.

(9) Consider the following matrix over $\mathbb{R}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

a) Find the characteristic polynomial for $\mathbf{A}$. b) What are the eigenvalues of $\mathbf{A}$ and what are their multiplicities? [Hint: $\lambda = 2$.] c) Determine whether $\mathbf{A}$ is diagonalizable? Explain your answer. If it is diagonalizable find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P^{-1}AP} = \mathbf{D}$.

(10) Circle your answer to the following true or false questions.

$\alpha$. True or False: $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

$\beta$. True or False: If $A$ is a $n \times n$ matrix and $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

$\gamma$. True or False: If $A$ and $B$ are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$.

$\delta$. True or False: Given the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^k$, $T$ is one-to-one if and only $A_T$ has a pivot position in every row.

$\epsilon$. True or False: A linearly independent set in a subspace $H$ is a basis for $H$.

$\zeta$. True or False: If $\mathcal{B} = \{v_1, \cdots, v_k\}$ is linearly independent set in $\mathbb{R}^k$, then $\mathcal{B}$ is a basis for $\mathbb{R}^k$.

$\eta$. True or False: A basis is a linearly independent set that is as large as possible.

$\theta$. True or False: If $v$ is a vector for which $Av = \lambda v$, then $v$ is an eigenvector for $\lambda$.

$\iota$. True or False: A matrix $A$ is not invertible if and only if $\lambda = 0$ is an eigenvalue of $A$.

$\kappa$. True or False: $A$ is diagonalizable if and only if $A$ has $n$ eigenvalues, counting multiplicities.

Bonus. Do this page last

(11) Let $\mathbf{A}$ be an $n \times n$ square matrix. Suppose that $\mathbf{A} = \mathbf{PDP^{-1}}$ for some invertible matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$. Prove by induction that for any natural number $n$ that $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P^{-1}}$. [Hint: Look at $\mathbf{A}^2$.]

(12) Let $V$ be a vector space. Prove that if $U_1$ and $U_2$ are subspaces of $V$, then $U_1 \cap U_2$ is also a subspace. (Be sure to define $U_1 \cap U_2$!)