Final Exam – MATH 332

Directions: Make sure to show all necessary work to receive full credit. If you need extra space please use the back of the sheet with appropriate labeling. Buena suerte.

1. [5 pts.] A vector space is a set which has an addition and a scalar multiplication by a field \( F \) such that

(i) (vi)

(ii) (vii)

(iii) (viii)

(iv)

(v)

2. [10 pts.] State the following definitions. \( A \) is an \( n \times n \)-matrix.

(i) The vector \( x \) is an eigenvector of \( A \) corresponding to the scalar \( \lambda \) if

(ii) The characteristic polynomial of \( A \) is

(iii) A function \( T : V \rightarrow W \) between 2 vector spaces \( V \) and \( W \) is called a linear transformation if

(iv) The matrices \( A \) and \( B \) are called similar if

(v) A set of vectors \( \{x_1, x_2, \ldots, x_n\} \) is said to be linearly dependent if

(vi) The square matrix is called upper triangular if
3. [4 pts.] Let $z = 5 - 3i$. Find $||z||$ and $z^{-1}$.

4. [7 pts.] Let $A$ be an $m \times m$ matrix. Suppose that $A = PDP^{-1}$ for some invertible matrix $P$ and diagonal matrix $D$. Prove by induction that for any natural number $n$ that $A^n = PD^nP^{-1}$.

5. [8 pts.] Let $V$ be a vector space. Prove that if $U_1$ and $U_2$ are subspaces of $V$ then $U_1 \cap U_2$ is also a subspace. (Be sure to define $U_1 \cap U_2$!)
6. [10 pts.] Let \( A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{bmatrix} \) Row reduce the matrix. What are the basic and free variables. Write the nul space of \( A \) as a span of vectors. What is the dimension of the nul space. Find the determinant of \( A \).

7. [5 pts.] Determine the value of \( h \) such that the matrix is the augmented matrix of a consistent linear system. \( A = \begin{bmatrix} 1 & h & -2 \\ -4 & 2 & 10 \end{bmatrix} \)

8. [3 pts.] Let \( S = \{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix} \} \) be 3 vectors in the plane. Is this set a linearly independent set? Explain.
9. [20 pts.] Consider the following matrix over $\mathbb{R}$.

$$
A = \begin{bmatrix}
4 & 4 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix}
$$

(a) Find the characteristic polynomial for $A$. What are the eigenvalues of $A$ and what are their multiplicities? (Graph it and find the roots... you may need to zoom in.)

(b) To each eigenvalue find the dimension of its corresponding eigenspace by finding a basis for the eigenspace. List a basis for each eigenspace.

(c) Is $A$ diagonalizable? Explain your answer. If it is find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. 

10. [10 pts.] Consider the following matrix over $\mathbb{Z}_3$.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

Find the characteristic polynomial over $\mathbb{Z}_3$. List the eigenvalues of $A$ if there are any.

11. [8 pts.] Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Write out $A^t$. Without doing any calculations determine the determinant and eigenvalues of $A$. Is $A$ invertible?
12. [1 pt. for each] State as many of the equivalent conditions listed in the invertible matrix theorem.

13. [10 pts.] Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $T(x, y, z) = (2x + 3y - z, x + z)$. Prove that $T$ is a linear transformation and find its standard matrix representation. Since there is collapsing what is its nul space?