Exam 3– MATH 3320 – Spring 2010

Directions: Make sure to show all necessary work to receive full credit. If you need extra space please use the back of the sheet with appropriate labeling. Good luck.

1. Let V be a vector space and suppose $v_1, v_2, \cdots, v_k \in V$. State the definition for the set $\{v_1, \cdots, v_k\}$ to be linearly independent. Be sure about your quantifiers.

2. Let V be a vector space. Define a basis for V.

3. Consider the map $S : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$S(x, y, z) = (3x - 2y + 4z, x + 3y, 5x - y - z, x - z).$$

a) Find A_S the standard matrix representation of S. Then find rref(A_S). Explain.

$$A_S = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \text{ and } \mathrm{rref}(A_S) = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix}$$

b) Are the column vectors of A_S linearly independent? Explain.

c) Do the column vectors of A_S span $\mathbb{R}^4$? Explain.

4. Consider the 3×5 matrix over $\mathbb{Z}_7$.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

a) Find a basis for the null space of A.

b) Find a basis for $\mathrm{Col}(A)$.

5. Let $W = \{(x, y) \in \mathbb{R}^2 | 0 \leq x^2 + y^2\}$. Determine whether W is a subspace of $\mathbb{R}^3$ or not. If it is prove it. Otherwise explain why not.

6. Let $V = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0\}$. Determine whether V is a subspace of $\mathbb{R}^3$ or not. If it is prove it. Otherwise explain why not.

7. Consider the 4×4 matrix over $\mathbb{Z}_5$.

$$B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & 0 & 2 & 3 \\ 4 & 1 & 3 & 3 \end{bmatrix}$$

Compute $\det(B)$. Show your work doing cofactor expansion– mention the row or column.

8. Let $T : V \to W$ be a linear transformation.

a) Define the null space of T, denoted $\mathrm{Nul}(T)$.

$$\mathrm{Nul}(T) = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}.$$

b) Prove that $\mathrm{Nul}(T)$ is a subspace of V.

9. Bonus!!! Suppose $T : V \to W$ is a linear transformation. Prove that $T(0_V) = 0_W$.

10. Bonus!!! Let V be a vector space and $v \in V$. What is $-v$?

11. Bonus!!! Circle your answer to the following questions.

α. True or False: $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

β. True or False: The vector $\mathbf{w} = (2, 1)$ is in the column space of $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$?

γ. True or False: If A and B are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$?.

δ. True or False: Given the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^k$, T is one-to-one if and only A_T has a pivot position in every row.

ϵ. True or False: A linearly independent set in a subspace H is a basis for H.

ζ. True or False: If $\mathcal{B} = \{v_1, \cdots, v_k\}$ is linearly independent set in $\mathbb{R}^k$, then $\mathcal{B}$ is a basis for $\mathbb{R}^k$.

η. True or False: A basis is a linearly independent set that is as large as possible.