1. Compute the determinant of the following $4 \times 4$ matrix. [Hint: look for 0s.]

$$H = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & -2 & 2 \end{pmatrix}.$$ 

2. Suppose $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$ and its characteristic polynomial is $c_A(x) = (x - 1)(x - 2)^2$.

(a) What are the eigenvalue(s) of $A$ and what are their algebraic multiplicities?

(b) Compute $2I_3 - A$.

(c) Find a basis for the eigenspace $E_2$ corresponding to the eigenvalue $\lambda = 2$.

(d) What is the geometric multiplicity of $\lambda = 2$?

(e) Is $A$ diagonalizable? Explain.
3. Let \( B = \begin{pmatrix} 4 & h \\ 2 & -3 \end{pmatrix} \).

For which value of \( h \) makes \( B \) an idempotent matrix? (If you do not know what that means you may ask me but I will write down that you did not know...and you will lose 25\% of the possible points.)

4. Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be the function defined by
\[
T( (x, y) ) = (x - 2y, 2x + y, y - x).
\]

(a) Prove that \( T \) is a linear transformation.

(b) Compute the matrix representation \( A_T \) of \( T \).
5. Let $V$ be the subspace of $\mathbb{R}^3$ spanned by the set \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \}. Find a basis for $V$ and determine the dimension of $V$.

6. Consider the following system of equations. Determine whether the system is consistent. If so, give a specific example of a solution.

\[
\begin{align*}
  x + 2y + 2z &= 2 \\
 -x + 2y &= 1 \\
 2y + z &= 1 
\end{align*}
\]
7. Let \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) be the linear transformation defined by \( T(v) = A \cdot v \) for each \( v \in \mathbb{R}^3 \) where

\[
A = \begin{pmatrix}
2 & 1 & 4 \\
3 & -1 & 11 \\
1 & 0 & 3 \\
4 & 3 & 6
\end{pmatrix}
\]
and \( \text{rref}(A) = \begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \)

(a) Are the column vectors of \( A \) linearly independent? Explain.

(b) Find a basis for \( \text{range}(T) \) and determine the dimension of \( \text{range}(T) \).

8. Suppose \( A \) is a \( 4 \times 4 \) matrix and \( c_A(x) \) factors into the nice form

\[ c_A(x) = (x + 1)(x - 2)(x - 1)^2. \]

You also know that \( \beta_1 = \{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix} \} \) is a basis for \( E_1 \), \( \beta_2 = \{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \} \) is a basis for \( E_2 \), and \( \beta_{-1} = \{ \begin{pmatrix} 2 \\ -2 \\ 3 \\ 1 \end{pmatrix} \} \) is a basis for \( E_{-1} \).

(a) Is \( A \) diagonalizable? If so, construct a \( P \) so that \( P^{-1}AP = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix} \).

(b) Is \( A \) invertible? Why or why not?
9. Let $A$ be a $m \times n$ matrix. Give a brief explanation of how to find a basis for the kernel of $A$, $\ker A$. For which previous problem in this test were you asked to do this?

10. Let $A$ be a $m \times n$ matrix. Give a brief explanation of how to find a basis for the column space, $\text{Col}(A)$. For which previous problem in this test were you asked to do this?

11. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation with matrix representation $A_T$. Give a brief explanation of how to determine whether the vector $\vec{b} \in \mathbb{R}^n$ is in the range of $T$, $\text{range}(T)$. For which previous problem in this test were you asked to do this?

12. Let $0 \leq \theta < 2\pi$. Explain why the matrix $E = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ is invertible. Find the inverse of the matrix.
13. Let \( V = \{ (x, y, z) \in \mathbb{R}^3 : 2x + 3y - z = 0 \} \). Prove that \( V \) is a subspace of \( \mathbb{R}^3 \). What is the dimension of \( V \). Find a basis.

14. True or False:
(a) \( \mathbb{R}^2 \) is a subspace of \( \mathbb{R}^3 \).
(b) Let \( T : \mathbb{R}^4 \to \mathbb{R}^2 \) be a linear transformation. It is not possible for \( T \) to be onto/surjective.
(c) An \( n \times n \) matrix is invertible if and only if it is singular.
(d) Let \( A \) be a \( m \times n \) matrix. If the kernel of \( A \) is trivial, then \( n \leq m \).
(e) If a subset of \( \mathbb{R}^n \) is linearly dependent, then the set contains more than \( n \) vectors.
(f) The vector equation \( 0 = \alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k \) has a unique solution if and only if the set of vectors \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \) is linearly independent.
(g) Let \( A \) be a \( n \times n \) matrix. The eigenvalues of \( A \) are the diagonal entries of \( A \).
(h) The permutation \( (1 \ 2 \ 3 \ 4)(5 \ 6) \) is even.
(i) Every singular \( n \times n \) matrix has an eigenvalue.
(j) If \( \vec{v} \) is a vector such that \( A\vec{v} = 2\vec{v} \), then \( \vec{v} \) is an eigenvector for \( A \).
(k) \( (1 \ 2 \ 4 \ 3) \circ (1 \ 2) = (2 \ 4 \ 3) \).
(l) If \( c_A(x) = (x^2 - 1)(x^2 - 4) \), then \( A \) is diagonalizable.
(m) Let \( \vec{0} \neq \vec{u} \in \mathbb{R}^3 \). The set \( \{ \vec{v}, \vec{u} \} \) is linearly independent if and only if \( \vec{u} \) is not on the line through the origin and \( \vec{v} \).
(n) The set \( \{ \vec{0} \} \) is a basis for the subspace \( \{ \vec{0} \} \).
15. Match the matrices below with the appropriate notation for an elementary matrix. The idea is the matrices below are on the left side of a matrix product $XA$ where you substitute the matrices below in for $X$.

(a) $F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(c) $H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(d) $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$

(e) $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & \beta \end{pmatrix}$

(f) $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & 0 & 1 \end{pmatrix}$

i. $D_3(\alpha) =$

ii. $S_{1,2} =$

iii. The matrix that places $\beta R_2 + \alpha R_3$ into row 2.

16. **Extra Credit.** Recall that $[-1,1] = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. Let $C([-1,1])$ be the set of continuous real-valued functions $f : [-1,1] \to \mathbb{R}$. $C([-1,1])$ is a vector space under the following addition and scalar multiplication: for all $f,g \in C([-1,1])$ and $\alpha \in \mathbb{R}$, and for all $x \in [-1,1]$

$$(f + g)(x) = f(x) + g(x) \text{ and } (\alpha \cdot f)(x) = \alpha \cdot f(x).$$

Recall that a function is called even if for all $x \in [-1,1]$, $f(-x) = f(x)$. Prove that the set

$W = \{f \in C([-1,1]) : f \text{ is even}\}$

is a subspace of $C([-1,1])$. 
17. Extra Credit How many $2 \times 2$ matrices with entries in $\mathbb{Z}_5$ are there? Of these how many are invertible. [Hint: Think of $A = (\vec{v}_1 \vec{v}_2)$ with non-zero vector $\vec{v}_1$ and for $A$ to be invertible what do you know about $\vec{v}_2$.]

18. Extra Credit This is over $\mathbb{Z}_5$. For what values of $h$ makes the following matrix diagonalizable?

$$A = \begin{pmatrix} 1 & 2 \\ 3 & h \end{pmatrix}$$