Name: ____

Final Exam – MAS 2103H Spring 2019

1. Compute the determinant of the following 4×4 matrix. [Hint: look for 0s.]

$$H = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & -2 & 2 \end{pmatrix}.$$

2. Suppose $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$ and its characteristic polynomial is $c_A(x) = (x-1)(x-2)^2$.

   (a) What are the eigenvalue(s) of A and what are their algebraic multiplicities?

   (b) Compute $2I_3 - A$.

   (c) Find a basis for the eigenspace E_2 corresponding to the eigenvalue $\lambda = 2$.

   (d) What is the geometric multiplicity of $\lambda = 2$?

   (e) Is A diagonalizable? Explain.

3. Let $B = \begin{pmatrix} 4 & h \\ 2 & -3 \end{pmatrix}$.

For which value of h makes B an idempotent matrix? (If you do not know what that means you may ask me but I will write down that you did not know...and you will lose 25% of the possible points.)

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$T(\ (x, y)\ ) = (x - 2y, 2x + y, y - x).$$

(a) Prove that T is a linear transformation.

(b) Compute the matrix representation A_T of T.

5. Let V be the subspace of $\mathbb{R}^3$ spanned by the set $\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\}$. Find a basis for V and determine the dimension of V.

6. Consider the following system of equations. Determine whether the system is consistent. If so, give a specific example of a solution.

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & 2 \\ -x & + & 2y & & & = & 1 \\ & & 2y & + & z & = & 1 \end{array}$$

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by $T(v) = A \cdot v$ for each $v \in \mathbb{R}^3$ where

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & -1 & 11 \\ 1 & 0 & 3 \\ 4 & 3 & 6 \end{pmatrix} \text{ and } \operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) Are the column vectors of A linearly independent? Explain.

(b) Find a basis for range(T) and determine the dimension of range(T).

8. Suppose A is a 4×4 matrix and $c_A(x)$ factors into the nice form

$$c_A(x) = (x+1)(x-2)(x-1)^2.$$

You also know that $\beta_1 = \{\begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \{\begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}\}$ is a basis for E_1, $\beta_2 = \{\begin{pmatrix} -1 \\ -1 \\ 3 \\ 0 \end{pmatrix}\}$ is a basis for E_2, and $\beta_{-1} = \{\begin{pmatrix} 2 \\ -2 \\ 3 \\ 1 \end{pmatrix}\}$ is a basis for E_{-1}.

(a) Is A diagonalizable? If so, construct a P so that $P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

(b) Is A invertible? Why or why not?

9. Let A be a $m \times n$ matrix. Give a brief explanation of how to find a basis for the kernel of A, $\ker A$. For which previous problem in this test were you asked to do this?

10. Let A be a $m \times n$ matrix. Give a brief explanation of how to find a basis for the column space, $\mathrm{Col}(A)$. For which previous problem in this test were you asked to do this?

11. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation with matrix representation A_T. Give a brief explanation of how to determine whether the vector $\vec{b} \in \mathbb{R}^n$ is in the range of T, $\mathrm{range}(T)$. For which previous problem in this test were you asked to do this?

12. Let $0 \leq \theta < 2\pi$. Explain why the matrix $E = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is invertible. Find the inverse of the matrix.

13. Let $V = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - z = 0\}$. Prove that V is a subspace of $\mathbb{R}^3$. What is the dimension of V. Find a basis.

14. True or False:
   (a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

   (b) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation. It is not possible for T to be onto/surjective.

   (c) An $n \times n$ matrix is invertible if and only if it is singular.

   (d) Let A be a $m \times n$ matrix. If the kernel of A is trivial, then $n \leq m$.

   (e) If a subset of $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

   (f) The vector equation $0 = \alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k$ has a unique solution if and only if the set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

   (g) Let A be a $n \times n$ matrix. The eigenvalues of A are the diagonal entries of A.

   (h) The permutation $(1\ 2\ 3\ 4)(5\ 6)$ is even.

   (i) Every singular $n \times n$ matrix has an eigenvalue.

   (j) If v is a vector such that $A\vec{v} = 2\vec{v}$, then $\vec{v}$ is an eigenvector for A.

   (k) $(1\ 2\ 4\ 3) \circ (1\ 2) = (2\ 4\ 3)$.

   (l) If $c_A(x) = (x^2 - 1)(x^2 - 4)$, then A is diagonalizable.

   (m) Let $\vec{0} \neq \vec{v} \in \mathbb{R}^3$. The set $\{\vec{v}, \vec{u}\}$ is linearly independent if and only if $\vec{u}$ is not on the line through the origin and $\vec{v}$.

   (n) The set $\{\vec{0}\}$ is a basis for the subspace $\{\vec{0}\}$.

15. Match the matrices below with the appropriate notation for an elementary matrix. The idea is the matrices below are on the left side of a matrix product XA where you substitute the matrices below in for X.

(a) $F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(c) $H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(d) $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$

(e) $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & \beta \end{pmatrix}$

(f) $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & 0 & 1 \end{pmatrix}$

i. $D_3(\alpha) =$

ii. $S_{1,2} =$

iii. The matrix that places $\beta R_2 + \alpha R_3$ into row 2.

16. **Extra Credit**. Recall that $[-1, 1] = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. Let $C([-1, 1])$ be the set of continuous real-valued functions $f : [-1, 1] \to \mathbb{R}$. $C([-1, 1])$ is a vector space under the following addition and scalar multiplication: for all $f, g \in C([-1, 1])$ and $\alpha \in \mathbb{R}$, and for all $x \in [-1, 1]$

$$(f + g)(x) = f(x) + g(x) \text{ and } (\alpha \cdot f)(x) = \alpha \cdot f(x).$$

Recall that a function is called *even* if for all $x \in [-1, 1]$, $f(-x) = f(x)$. Prove that the set

$$W = \{f \in C([-1, 1]) : f \text{ is even }\}$$

is a subspace of $C([-1, 1])$.

17. **Extra Credit** How many 2×2 matrices with entries in $\mathbb{Z}_5$ are there? Of these how many are invertible. [Hint: Think of $A = (\vec{v}_1 \vec{v}_2)$ with non-zero vector $\vec{v}_1$ and for A to be invertible what do you know about $\vec{v}_2$.]

18. **Extra Credit** This is over $\mathbb{Z}_5$. For what values of h makes the following matrix diagonalizable?

$$A = \begin{pmatrix} 1 & 2 \\ 3 & h \end{pmatrix}$$