Exam 1– MAS 2103H – Spring 2019

Directions: Make sure to show all necessary work to receive full credit. If you need extra space please use the extra blank sheet with appropriate labeling.

1. True/False
   (a) True or False: If for some matrix A, and some vectors $\vec{x}, \vec{b}$, we have $A\vec{x} = \vec{b}$, then $\vec{b}$ is in the span of the column vectors of A.
   (b) True or False: In $\mathbb{Z}_5$, every element has a multiplicative inverse.
   (c) True or False. The homogeneous equation $\mathbf{A}\vec{x} = \vec{0}$ always has a solution.
   (d) True or False: A $m \times n$ matrix has n columns.
   (e) True or False: In $\mathbb{Z}_{12}$, the number 2 has a multiplicative inverse.
   (f) True or False: The determinant of a square matrix A is nonzero if and only if the equation $A\vec{x} = \vec{0}$ has a unique solution.

2. Definitions
   (a) Suppose V and W are $\mathbb{F}$-vector spaces. State what it means for the function $T : V \to W$ to be a linear transformation.
   (b) State the definition of Span$(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k})$.
   (c) Suppose $T : V \to W$ is a linear transformation between two $\mathbb{F}$-vector spaces. Define the kernel of T.

3. Compute AB and A^t where $A = \begin{bmatrix} -1 & 3 & 0 & 2 \\ 4 & -1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 0 \\ 1 & 1 & 2 \\ 3 & -3 & 0 \end{bmatrix}$.

4. Consider the following 3×3 matrix over $\mathbb{Z}_5$.

$$C = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 3 \\ 4 & 2 & 0 \end{bmatrix}$$

(a) Row reduce the matrix over $\mathbb{Z}_5$.

(b) Find a basis for the kernel of C.

(c) Are the column vectors (of C) $v_1 = (2, 1, 4), v_2 = (2, 3, 2), v_3 = (4, 3, 0)$ a basis for $\mathbb{Z}_5^3$?

5. Consider the matrix over $\mathbb{R}$ and its row echelon form.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}; \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Circle the pivot positions in rref(A). How many free variables are there? What are they?

(b) Find a basis for the kernel of A.

(c) Do the column vectors of A span all of $\mathbb{R}^4$? Explain.

6. Let $V = \mathbb{R}^n$ and let $S = \{v_1, \ldots, v_k\}$ be a collection of k-many vectors in V. Explain what you would do to determine whether the set S is linearly independent.

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x - y - z, y - z, z - x).$$

(a) Prove that T is a linear transformation.

(b) Find the matrix representation A_T of T.

(c) Row reduce A_T.

(d) Compute the determinant of A_T.

(e) Is the vector $\vec{v} = (3, 4, 6)$ in the span of the column vectors of A_T?