Directions: Make sure to show all necessary work to receive full credit. If you need extra space please use the extra blank sheet with appropriate labeling.

1. True/False
   (a) True or False: If for some matrix \( A \), and some vectors \( \vec{x}, \vec{b} \), we have \( A\vec{x} = \vec{b} \), then \( \vec{b} \) is in the span of the column vectors of \( A \).

   (b) True or False: In \( \mathbb{Z}_5 \), every element has a multiplicative inverse.

   (c) True or False. The homogeneous equation \( A\vec{x} = \vec{0} \) always has a solution.

   (d) True or False: A \( m \times n \) matrix has \( n \) columns.

   (e) True or False: In \( \mathbb{Z}_{12} \), the number 2 has a multiplicative inverse.

   (f) True or False: The determinant of a square matrix \( A \) is nonzero if and only if the equation \( A\vec{x} = \vec{0} \) has a unique solution.

2. Definitions
   (a) Suppose \( V \) and \( W \) are \( \mathbb{F} \)-vector spaces. State what it means for the function \( T: V \rightarrow W \) to be a linear transformation.

   (b) State the definition of \( \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) \).

   (c) Suppose \( T: V \rightarrow W \) is a linear transformation between two \( \mathbb{F} \)-vector spaces. Define the kernel of \( T \).
3. Compute $AB$ and $A^t$ where $A = \begin{bmatrix} -1 & 3 & 0 & 2 \\ 4 & -1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 0 \\ 1 & 1 & 2 \\ 3 & -3 & 0 \end{bmatrix}$.

4. Consider the following $3 \times 3$ matrix over $\mathbb{Z}_5$.

$$C = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 3 \\ 4 & 2 & 0 \end{bmatrix}$$

(a) Row reduce the matrix over $\mathbb{Z}_5$.

(b) Find a basis for the kernel of $C$.

(c) Are the column vectors (of $C$) $v_1 = (2, 1, 4), v_2 = (2, 3, 2), v_3 = (4, 3, 0)$ a basis for $\mathbb{Z}_5^3$?
5. Consider the matrix over \( \mathbb{R} \) and its row echelon form.

\[
A = \begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}; \quad \text{rref}(A) = \begin{bmatrix}
1 & 0 & -3 & 0 & 5 \\
0 & 1 & 2 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(a) Circle the pivot positions in \( \text{rref}(A) \). How many free variables are there? What are they?

(b) Find a basis for the kernel of \( A \).

(c) Do the column vectors of \( A \) span all of \( \mathbb{R}^4 \)? Explain.

6. Let \( V = \mathbb{R}^n \) and let \( S = \{v_1, \ldots, v_k\} \) be a collection of \( k \)-many vectors in \( V \). Explain what you would do to determine whether the set \( S \) is linearly independent.
7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
\[ T(x, y, z) = (x - y - z, y - z, z - x). \]
(a) Prove that $T$ is a linear transformation.

(b) Find the matrix representation $A_T$ of $T$.

(c) Row reduce $A_T$.

(d) Compute the determinant of $A_T$.

(e) Is the vector $\vec{v} = (3, 4, 6)$ in the span of the column vectors of $A_T$?