MAD 2104H
HW #2
Due November 1st, 2010.

1. Use Mathematical Induction to prove that the following formula holds for all positive $n \in \mathbb{N}$.
$$\sum_{i=1}^{n} i2^i = 2 + (n-1)2^{n+1}.$$
2. Consider the recursively defined sequence given by $f_0 = 1 = f_1$ and then for all natural numbers $n \in \mathbb{N}$, $f_{n+2} = f_{n+1} + f_n$. Observe that $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, and so on. (This sequence is called the *Fibonacci sequence*.) Prove by mathematical induction that for each natural number $n \geq 2$, gcd(f_n,f_{n+1})=1. [Hint: use that the gcd is the least positive $\mathbb{Z}$-linear combination.]

3. Let X be a set and recall that each element of X is again a set. Define a relation on X as follows
$$R = \{(x_1, x_2) \in X \times X : x_1 \subseteq x_2\}.$$
Prove that R is a partial order. Then give a specific example of a set X for which the relation on X is not a linear order.

4. Consider the following relation defined on $\mathbb{Z}^2$. For $(a, b), (c, d) \in \mathbb{Z}$ we define $(a, b) \leq (c, d)$ if either of the mutually exclusive events occur: 1) $a < c$, or 2) $a = c$ and $b \leq d$. Prove that $\leq$ is a partial order. Moreover $\leq$ is a linear order on $\mathbb{Z}^2$.

5. Supply a Fitch Proof for the following arguments.

(1) | $\underline{\forall x\ (\mathrm{P}(x) \rightarrow \mathrm{Q}(x))}$

| $\forall x\ (\neg\mathrm{Q}(x) \rightarrow \neg\mathrm{P}(x))$

(2) | $\forall y\ [\mathrm{Cube}(y) \vee \mathrm{Dodec}(y)]$
| $\forall x\ [\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$
| $\underline{\exists x\ \neg\mathrm{Large}(x)}$

| $\exists x\ \mathrm{Dodec}(x)$

(3) | $\underline{\exists x \forall y\ (\mathrm{Cube}(y) \leftrightarrow y = x)}$

| $\exists x\ (\mathrm{Cube}(x) \wedge \forall y\ (\mathrm{Cube}(y) \rightarrow y = x))$

6. Recall that $[n] = \{1, 2, \cdots, n\}$. What is the largest number of subsets we can select from $[n]$ such that any two selected subsets have at least one element in common.

7. Angel and Kayleigh play with dice. They throw four dice at the same time. If at least one of the four dice shows a six, then Angel wins. If not, then Kayleigh wins. Who has the greater chance of winning?

8. How many $n \times n$ matrices are there whose entries are 0 or 1 and in which each row and column has an even sum?