Review of MAC 2311 for Fall 2019

1. Functions
   (a) domain, range, inverse, vertical line test, horizontal line test,
   (b) Polynomials: $f(x) = a_n x^n + \ldots + a_1 x + a_0$; leading coefficient, degree of a polynomial
   (c) Rational Functions: $h(x) = \frac{p(x)}{q(x)}$ for polynomials $p(x), q(x)$.
   (d) Trigonometric Functions: $\sin x, \cos x, \tan x, \sec x, \cot x, \csc x$.
   (e) Exponential Functions: $f(x) = e^x$.
   (f) Logarithmic Functions: $f(x) = \ln x$.
   (g) Algebraic Functions: $f(x) = x^r$ for a real number $r \in \mathbb{R}$.
2. Definition of Limit:
$$\lim_{x\to a} f(x) = L$$
   if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.
3. Limits: calculations of limits, algebraic manipulations, Squeezing Theorem.
4. One-sided limits, Two-sided limits
5. Limits at infinity: Horizontal Asymptotes
6. If $f(x), g(x)$ are polynomials, then $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ exists if $\deg f(x) \le \deg g(x)$ and is the ratio of the leading coefficients. Otherwise it is $\pm\infty$.
7. L'Hopital's Rule: Suppose $f(a) = g(a) = 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$
8. Continuity: f is continuous at a if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$.
9. Discontinuities: removable discontinuities, jump discontinuities, vertical asymptotes, none of these (e.g. $f(x) = \sin(\frac{1}{x})$).
10. Intermediate Value Theorem for continuous functions: if f is continuous on the closed interval $[a, b]$ then for every z between $f(a)$ and $f(b)$ there is a $c \in [a, b]$ such that $f(c) = z$. In particular, if $f(a) > 0$ and $f(b) < 0$ then there is a zero of f between a and b.
11. If f is continuous on the closed interval $[a, b]$ then f attains its maximum and minimum values.
12. Rates of Change: average rate of change (slope of secant line) versus instantaneous rate of change (slope of tangent line).
13. Derivative with respect to a variable, e.g. $\frac{d}{dx}$, $f'(x)$
14. Derivative as a difference quotient or as limit of slopes of secant lines.
$$f'(a) = \lim_{x\to a} \frac{f(x) - f(x)}{x - a}.$$
15. Derivative.
$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
16. If f is differentiable at a, then f is continuous at a. The converse is false, e.g. $f(x) = |x|$ at $x = 0$.
17. Mean Value Theorem for differentiable functions: If f is differentiable on $[a, b]$ then there is some $c \in [a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
18. What the first derivative tells us about the graph of $f(x)$: critical points, increasing, decreasing, relative extrema.
19. What the second derivative tells us about the graph of $f(x)$: convexity.

20. An object is thrown in the air. Its height (in feet) is determined by the function

$$h(t) = -16t^2 + v_0 t + h_0$$

while its velocity is given by $v(t) = -32t + v_0$ and $a(t) = -32$. $v_0 = v(0)$ is its initial velocity and $h_0 = h(0)$ is its initial height. From some initial conditions you should be able to find 1) when the object hits the ground, 2) its maximum height, 3) the velocity or speed when it hits the ground, 4) how long it is in the air.

21. *Speed* is $|v(t)|$ (absolute value of velocity).

22. A particle moves along the x-axis with position function $p(t)$. Its velocity is given by $p'(t) = v(t)$. it acceleration is given by $p''(t) = v'(t) = a(t)$.

23. Rules of Differentiation

    (a) Sum of Derivatives is Derivative of Sums

    (b) Power Rule: $\frac{d}{dx} ax^n = nax^{n-1}$

    (c) Product Rule: $(uv)' = u'v + v'u$

    (d) Quotient Rule:$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$

    (e) Chain Rule: $(f \circ g)'(x) = f'(g(x))g'(x)$.

24. Implicit Differentiation

25. Related Rates: most examples involve: triangles, circles, cylinder, squares, areas, volumes,

26. Integration, Areas, Anti-Derivatives, Definite Integrals, Indefinite Integrals

27. Riemann Sums: approximating areas by rectangles (or trapezoids).

28. Given: $f(x)$ continuous on $[a, b]$. The (net) area under the curve is the definite integral $\int_a^b f(t)dt$.

29. To approximate we slice up $[a, b]$ into n-equal parts. Each part has length $\Delta x = \frac{b-a}{n}$. We construct n rectangles by using the points $a_0 = a, a_1 = a + \Delta x, \ldots, a_k = a + k\Delta x, \ldots, a_n = b$. Then the area of the rectangles is given by

$$LRS : \sum_{i=0}^{n-1} f(a_i)\Delta x$$

$$RRS : \sum_{i=1}^{n} f(a_i)\Delta x$$

$$\text{Midpoint}RS : \sum_{i=0}^{n-1} f(\frac{a_i + a_{i+1}}{2})\Delta x$$

If we use trapezoids then the approximate area is given by

$$TRS : \sum_{i=0}^{n-1} (\frac{f(a_i) + f(a_{i+1})}{2})\Delta x$$

30. Given: $f(x)$ continuous on $[a, b]$.

$$\int_a^b f(t)dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a_i)\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\frac{a_i + a_{i+1}}{2})\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} (\frac{f(a_i) + f(a_{i+1})}{2})\Delta x$$

31. If you know that f is increasing over $[a, b]$ what does this tell you about comparing $\displaystyle\int_a^b f(t)dt$, RRS, LRS? What is f is decreasing.

32. If you know that f is concave up over $[a, b]$ what does this tell you about comparing $\displaystyle\int_a^b f(t)dt$ and TRS? What about if f is concave down.

33. You should be able to use the sum formulas to calculate $\displaystyle\int_a^b f(t)dt$: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$; $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, etc.

34. (First) Fundamental Theorem of Calculus: let f be a continuous function on the closed interval $[a, b]$. Then the area function:
$$A(x) = \int_a^x f(t)dt$$
is a differentiable function for which $A'(x) = f(x)$. Therefore, f has an anti-derivative.

35. (Second) Fundamental Theorem of Calculus: let f be a continuous function on the closed interval $[a, b]$ and suppose F' is an anti-derivative of f. Then $\int_a^b f(t)dt = F(b) - F(a)$.

36.
$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

37.
$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f((u(x))u'(x).$$

38.
$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f((u(x))u'(x) - f((v(x))v'(x).$$

39. If you know the velocity function $v(t)$ of a particle moving along the x-axis then its total displacement (or net change in position) from time t_0 to time t_1 is given by
$$\int_{t_0}^{t_1} v(t)dt.$$

40. If you know the velocity function $v(t)$ of a particle moving along the x-axis then its total distance traveled from time t_0 to time t_1 is given by
$$\int_{t_0}^{t_1} |v(t)|dt.$$

41. The Mean Value Theorem for integrals: If $f(x)$is continuous over an interval $[a, b]$, then there is at least ne point $c \in [a, b]$ such that
$$f(c) = \int_a^b f(x)dx.$$

42. The *average value of f over* $[a, b]$ is $\displaystyle\frac{1}{b-a}\int_a^b f(t)dt$.

43. $\ln x = \int_1^x \frac{1}{t} dt$

44. The inverse function of $f(x) = \ln x$ is the exponential functions $f^{-1}(x) = e^x$.

45. Two functions f, g are called *inverses* of each other if $f(g(x)) = x$ and $g(f(x)) = x$.

46. Rules of Integrations:

(a) Sum Rule: $\int (f(t) + g(t))dt = \int f(t)dt + \int g(t)dt$.

(b) Constant Rule: $\int cf(t)dt = c \int f(t)dt$.

(c) Power Rule: $\int x^k dx = \frac{x^{k+1}}{k+1} + C$ (for $k \neq -1$).

(d) $\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + C$.

(e) $\int \cos(kx)dx = \frac{1}{k}\sin(kx) + C$.

(f) $\int \sec^2(kx)dx = \frac{1}{k}\tan x + C$.

(g) $\int \sec(kx)\tan(kx)dx = \frac{1}{k}\sec(kx) + C$.

(h) $\int \frac{1}{x}dx = \ln|x| + C$.

(i) $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$.

(j) Substitution: $\int f'(u(x))u'(x)dx = f(u(x)) + C$.

(k) $\int \tan x dx = \ln|\cos x| + C$.

(l) $\int \sin x \cos x dx = \frac{1}{2}\sin^2 x + C = -\frac{1}{2}\cos^2 x + C$.