Name: ________________________________

Final Exam – MAC 2312 – Fall 2021

Directions: For the multiple choice part make sure you clearly label your answer. Also, be sure to include your work as I might give partial credit. On the free response part make sure to show all necessary work to receive full credit. If you need extra space please use the extra blank sheets with appropriate labeling.

1. Evaluate the following integral.

$$\int \frac{1}{x^2 - 4x + 8}\, dx$$

(A) $\frac{1}{2}\ln|x-4| + \frac{1}{2}\ln|x+2| + C$

(B) $\frac{1}{2}\ln|x-4| + \frac{1}{2}\ln|x-2| + C$

(C) $\frac{1}{2}\arctan(\frac{x-2}{2}) + C$

(D) $\arctan(\frac{x-2}{2}) + C$

(E) $\frac{1}{2}\ln|x+4| - \frac{1}{2}\ln|x-22| + C$

2. The coefficient of x^4 in the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$ is

(A) $-\frac{3}{8}$ (B) $\frac{3}{8}$ (C) 0 (D) $\frac{3}{4}$ (E) $\frac{3}{32}$

3. Which of the following series converges conditionally?

I. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ II. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\ln n}$ III. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$

(A) I. only (B) I. and II. only (C) III. only (D) II. only

(E) II. and III. only (F) I. and III. only (G) I., II., and III.

4. The area of the region bounded by the curves $y = 2x$ and $y = x^2$ is

(A) $\frac{2}{3}$ (B) $\frac{8}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{6}$ (E) $\frac{11}{6}$

5. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = -1$.

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) 2π (D) $\frac{17}{3}$ (E) π

6. Evaluate the following integral $\int_0^3 \sqrt{9-x^2}\,dx$.

(A) $\frac{3\pi}{2}$ (B) $\frac{9\pi}{2}$ (C) $3\pi^2$ (D) $\frac{9\pi}{4}$ (E) 9π

7. Find the sum $\sum_{n=2}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$

(A) $\frac{3}{2}$ (B) $\frac{7}{12}$ (C) $\frac{5}{6}$ (D) $\frac{5}{2}$ (E) 1

8. Compute the following limit: $\lim_{n\to\infty} n\sin\left(\frac{\pi}{n}\right)$

(A) π (B) ∞ (C) 0 (D) $\frac{1}{\pi}$ (E) e^{π} (F) diverges and not to ∞

9. The area between the graphs of $y=x^2$ and $y=2x$ is revolved around the y-axis. If the method of cylindrical shells is used, the integral representing the volume of the resulting solid is

(A) $\int_0^4 2\pi x(x^4-4x^2)\,dx$ (B) $\int_0^4 2\pi x(x^2-2x)\,dx$

(C) $\int_0^2 2\pi x(2x-x^2)\,dx$ (D) $\int_0^1 \pi(x^4-4x^2)\,dx$

(E) $\int_0^2 2\pi(x+1)(x^2-2x)\,dx$

10. Find the interval of convergence for the Taylor series $\sum_{n=1}^{\infty}\frac{3^n}{n!}(x-5)^n$.

(A) $(-\frac{1}{3},\frac{1}{3})$ (B) $(5-\frac{e}{3},5+\frac{e}{3})$ (C) $(5-\frac{e}{3},5+\frac{e}{3}]$

(D) $(5-\frac{1}{3},5+\frac{1}{3})$ (E) $(-\infty,\infty)$

11. What does the integral $\int \frac{x^2}{\sqrt{x^2+25}}\,dx$ become after a trigonometric substitution?

(A) $25\int(\tan^2\theta)(\sec\theta)\,d\theta$ (B) $5\int(\tan^2\theta)(\sec\theta)\,d\theta$ (C) $25\int\sin^2\theta\,d\theta$

(D) $25\int\frac{\tan^2\theta}{\sec\theta}\,d\theta$ (E) $5\int\frac{\tan^2\theta}{\sec\theta}\,d\theta$

12. Find the interval of convergence for the Taylor series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n^5 3^n}$.

(A) $[2,4]$ (B) $(-3,3)$ (C) $(-\infty,\infty)$ (D) [0,6] (E) (0,6)

13. Which of the following are true?
I. If a series absolutely converges, then it converges.
II. If a series converges, then it absolutely converges.
III. If a series diverges, then it cannot converge conditionally.

(A) Only I. (B) Only II. (C) Only I. and II.

(D) Only II. and III. (E) Only I. and III. (F) I., II., and III.

14. The base of a solid is the region in the plane above the x-axis and below the curve $y=\sqrt{1-x^2}$. The cross sections of the solid perpendicular to the x-axis axis are squares. Find the volume of the solid.

(A) $\frac{8}{3}$ (B) $\frac{4}{3}$ (C) $\frac{2}{3}$ (D) $\frac{\pi}{2}$ (E) $\frac{\pi}{4}$

15. $\int xe^{2x}dx =$

(A) $\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+C$ (B) $\frac{xe^{2x}}{2}-\frac{e^{2x}}{2}+C$ (C) $\frac{xe^{2x}}{2}+\frac{e^{2x}}{4}+C$

(D) $2xe^{2x}-4e^{2x}+C$ (E) $\frac{x^2e^{2x}}{4}+C$

16. The region bounded by $y=e^x$, $y=1$, and $x=2$ is revolved about the x-axis. If the washer method is used, the volume of the solid generated is given by the integral

(A) $\pi\int_0^2 e^{2x}\,dx$ (B) $2\pi\int_1^2(1-e^x)\,dx$ (C) $\pi\int_0^2(e^{2x}-1)\,dx$

(D) $\pi\int_0^2(e^x-1)^2\,dx$ (E) $\pi\int_0^2(1-e^{2x})\,dx$

17. Compute the sum $\sum_{n=1}^{\infty} \frac{1}{3^n}$

(A) π (B) $\frac{1}{3}$ (C) $\frac{3}{2}$ (D) $\frac{1}{2}$ (E) 2

18. Compute $\int -9x\cos(5x)\ dx$.

(A) $-\frac{9}{25}\cos(5x) - \frac{9}{5}x\sin(5x) + C$

(B) $-\frac{9}{5}\cos(5x) - 9x\sin(5x) + C$

(C) $-\frac{9}{25}\cos(5x) - \frac{9}{5}\sin(5x) + C$

(D) $-\frac{9}{5}\cos(5x) - \frac{9}{5}x\sin(5x) + C$

(E) $\frac{9}{25}\cos(5x) - \frac{9}{5}x\sin(5x) + C$

19. Find the limit:

$$\lim_{n\to\infty} \left(\frac{2n+3}{2n+1}\right)^{2n}$$

(A) $e^{\frac{1}{2}}$ (B) 1 (C) e^{-2} (D) e (E) e^{-1} (F) e^2

20. If $\int_1^{\infty} \frac{dx}{x^p}$ converges, then which of the following must be true?

(A) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

(B) $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges

(C) $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges

(D) $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges

(E) $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ converges

21. Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x. What is $f'(1)$?

(a) a_0 (b) a_1 (c) 0 (d) $\sum_{n=1}^{\infty} n a_n$ (e) $\sum_{n=1}^{\infty} n a_n^{n-1}$

22. Find the arc length of the curve $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ over the closed interval $[2, 3]$. [Hint: complete the square.]

23. Let $f(x) = x\cos(x^2)$. Compute the second degree Taylor polynomial $P_2(x)$ centered at $x = 0$. [Hint: Compute $f'(x), f''(x)$.]

24. Compute the improper integral. [Hint: Consider a trig sub and a trig identity.]

$$\int_1^\infty \frac{1}{x^2(x^2+1)}\,dx$$

25. Use Partial Fractions to find the indefinite integral

$$\int \frac{2x-2}{(x+1)(x-2)}$$

26. Determine whether each of the following series converges or diverges. Circle your answer.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^n+1}$ converges / diverges

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ converges / diverges

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(\ln n)^n}$ converges / diverges

d. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges / diverges

e. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6}{7}\right)^n$ converges / diverges

f. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2^n}{n^n}\right)^2$ converges / diverges

g. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$ converges / diverges

h. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ converges / diverges

i. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$ converges / diverges

j. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$ converges / diverges

k. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 5 \cdots (3n-1)}$ conv / div

l. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges / diverges

27. The Maclaurin series for a function $f(x)$ is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n-1}}{n}x^n$$

(a) Use the Ratio Test to find the radius of convergence of $f(x)$.

(b) Find the Interval of Convergence of the power series.

(c) Write the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(d) What is the interval of convergence for $f'(x)$.

(e) Compute $\lim_{x\to 0} \dfrac{1 - f'(x)}{x}$.

28. **(Extra Credit)** Which is (most) true? The series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is

(A) convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(B) divergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(C) divergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

(D) convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

(E) convergent by integral test

29. **(Extra Credit)** Which of the following is a power series expansion for $\frac{x^2}{1-x^2}$

(A) $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$

(B) $\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + x^5 + \cdots$

(C) $\sum_{n=0}^{\infty} (n+1)x^{n+2} = x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots$

(D) $\sum_{n=0}^{\infty} x^{2(n+1)} = x^2 + x^4 + x^6 + x^8 + \cdots$

(E) $\sum_{n=0}^{\infty} (-1)^n x^{2(n+1)} = x^2 - x^4 + x^6 - x^8 + \cdots$

30. **(Extra Credit)** Find the radius of convergence and interval of convergence of the following power series. $\sum_{n=1}^{\infty} \frac{(x+5)^{3n}}{8^n}$

Formulas

(a) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n;\ I = (-1,1)$

(b) $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!};\ I = (-\infty,\infty)$

(c) $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!};\ I = (-\infty,\infty)$

(d) $\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!};\ I = (-\infty,\infty)$

(e) $\ln(1+x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^n}{n};\ I = (-1,1]$

(f) $\arctan x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1};\ I = [-1,1]$

(g) $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n;\ I = (-1,1)$

(h) $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

(i) $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

(j) $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

(k) $\dfrac{d}{dx}(\text{sech}^{-1} x) = \dfrac{-1}{x\sqrt{1+x^2}}$

(l) $\displaystyle\int \csc x\ dx = -\cot x + C$

(m) $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + C$

(n) $\displaystyle\int \sec^3 x\ dx = \frac{1}{2}(\ln|\sec x + \tan x| + \tan x \sec x) + C$

(o) $\displaystyle\int \tan x\ dx = \ln|\sec x| + C$

(p) $\displaystyle\int u\ dv = uv - \int v\ du$

(q) $\sin^2 x = \frac{1-\cos(2x)}{2} \quad \cos^2 x = \frac{1+\cos(2x)}{2}$

(r) $\cos^2 x + \sin^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x$

(s) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

(t) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

(u) $\sin(2\theta) = 2\sin\theta\cos\theta$

(v) $\cos(2\theta) = 1 - 2\sin^2\theta$

(w) $\binom{r}{n} = \frac{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}{n!}$

(x) $\cosh x = \dfrac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh x = \sinh x$

(y) $\sinh x = \dfrac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh x = \cosh x$

Trig Substitutions:

i. Use $x = a\sin\theta$ for expressions with $\sqrt{a^2-x^2}$

ii. Use $x = a\tan\theta$ for expressions with $\sqrt{a^2+x^2}$

iii. Use $x = a\sec\theta$ for expressions with $\sqrt{x^2-a^2}$

<u>Volumes and Surface Areas:</u>

(a) Cross-sections: $\int_a^b A(x)\ dx$

(b) Cylindrical Shells: $\int_a^b 2\pi x f(x)\ dx$

(c) Arc length: $\int_a^b \sqrt{1+f'(x)^2}\ dx$

(d) Disc/Washer: $\int_a^b \pi\big(f(x)^2 - g(x)^2\big)\ dx$

(e) Lateral Surface Area: $\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\ dx$