1. Sets, Relations, and Functions

To be successful in this course you should, by now, have a strong grasp of logic as well as a good understanding of the model of Set Theory. Recall this is the first order logic whose objects are sets and which has two binary predicates: =, ∈. The following symbols denote the standard boolean operators of disjunction, conjunction, conditional, negation, and bi-conditional: ∨, ∧, →, ¬, ↔. Our main symbols for quantifiers are: ∀ (universal) and ∃ (existential).

The sentence $x \in A$ means that $x$ is a member of $A$. We will write $x \not\in A$ instead of $\neg(x \in A)$.

**Definition 1.1.** Suppose $A$ and $B$ are sets. We write $A \subseteq B$ (and say $A$ is a subset of $B$) if every element of $A$ is an element of $B$. This means that $\forall x(x \in A \rightarrow x \in B)$.

We now recall our axioms of Set Theory.

**Axiom of Extensionality** Two sets $A$ and $B$ are equal if and only if they have the same elements, i.e. $A \subseteq B$ and $B \subseteq A$.

**Axiom of Set Existence** There exists a set which does not contain any elements. This set is denoted by $\emptyset$ and called the *empty set*.

**Axiom of Specification** To every set $A$ and to every well-defined formula $S(x)$ there corresponds a set $B$ whose elements are exactly those elements $x \in A$ for which $S(x)$ holds. This set is denoted by $\{x \in A : S(x)\}$.

**Axiom of Unions** We will assume that every collection of sets (which contains at least one set) can be written as $\{A_i\}_{i \in I}$ where $I$ denotes some index set. The axiom of union says that there is a set, denoted $\bigcup_{i \in I} A_i$ which consists of precisely those elements that belong to at least one of the $A_i$. When the index set is finite, say $i_1, \cdots, i_n$ we instead write $A_{i_1} \cup \cdots \cup A_{i_n}$.

**Proposition 1.2.** For any sets $A, B, C$

1. $A \cup B = B \cup A$.
2. $A \cup (B \cup C) = (A \cup B) \cup C$.
3. $A \cup A = A$.
4. $A \subseteq B$ if and only if $A \cup B = B$.

**Lemma 1.3.** Let $\{A_i\}_{i \in I}$ be a family of sets. There is a set whose elements are precisely the elements that belong to $A_i$ for all $i \in I$.

**Remark 1.4.** The set given by Lemma 1.3 is called the *intersection* of the sets and is denoted by $\bigcap_{i \in I} A_i$. When $I$ is a finite set, say $i_1, \cdots, i_n$ then we instead write $A_{i_1} \cap \cdots \cap A_{i_n}$. If two sets, say $A$ and $B$ satisfy $A \cap B = \emptyset$, then $A$ and $B$ are said to be *disjoint sets.*
**Axiom of Power Sets** For any set $A$ there exists a set whose elements are precisely the subsets of $A$. This set is called the *power set of $A$* and is denoted by $\mathcal{P}(A)$.

**Axiom of Ordered Pairs** Let $A, B$ be sets. The *cartesian Product of $A$ and $B$* is the set of ordered pairs of the form $(a, b)$ for $a \in A, b \in B$; the set is denoted by $A \times B$. When $A = B$ we instead write $A^2$ (instead of $A \times A$).

Ordered pairs satisfy the property that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

2. **Real Numbers**

We let $\mathbb{R}$ denote the set of real numbers. The set of positive real numbers is denoted by $\mathbb{R}^+$ and is the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

**Definition 2.1.** For a set $S \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, we call $u$ an *upper bound of $S$* if $u \geq x$ for every $x \in S$. Not every subset of $\mathbb{R}$ possesses an upper bound. If there is an upper bound for $S$, then we say that $S$ is *bounded above*. A *lower bound* and a *bounded below* set are defined analogously.

Let $S \subseteq \mathbb{R}$ and let $b \in \mathbb{R}$. We say that $b$ is a *least upper bound for $S$* if it satisfies the following two properties:

i. $b$ is an upper bound of $S$,

ii. For any other upper bound of $S$, say $u \in \mathbb{R}$, $b \leq u$.

A greatest lower bound for a set is defined analogously.

**Lemma 2.2.** Suppose $S \subseteq \mathbb{R}$. If $S$ has a least upper bound, then it is unique. Similarly, for greatest lower bound. (So we can use the definite article the instead of the indefinite article a.) We denote the least upper bound (resp. greatest lower bound) if $S$, by $\text{lub}(S)$ (resp. $\text{glb}(S)$).

**Question 2.3.** Is the empty set bounded above (or below)? If so, is there a lub (or glb)?

**Axiom of Real Numbers** We assume that the set of real numbers is *Dedekind Complete*, that is, whenever a set is bounded above (resp. below), then the set in question possesses a least upper bound (resp. greatest lower bound). This statement is actually a theorem however we shall simply assume that $\mathbb{R}$ is Dedekind Complete. [Named after Richard Dedekind c. 1831-1916 who developed the foundational theory of the real numbers.]

**Proposition 2.4.** Let $\emptyset \neq B \subseteq A \subseteq \mathbb{R}$. If $A$ is bounded above, then so is $B$ and $\text{lub}(B) \leq \text{lub}(A)$. Analogously for $\text{glb}$.

**Definition 2.5.** Consider the following function defined on $\mathbb{R}$: for $x \in \mathbb{R}$

$$|x| = \begin{cases} x & x \in \mathbb{R}^+ \ 
-x & x \in \mathbb{R} \setminus \mathbb{R}^+ \end{cases}$$

**Proposition 2.6** (R. Rohan). *The absolute value function has the following properties for all $x, y \in \mathbb{R}$:*

1. $0 \leq |x|$,
2. $0 = |x|$ if and only if $x = 0$,
3. $|x| = |-x|$,
4. $x \leq |x|$,
(5) $|xy| = |x||y|$,  
(6) $|x + y| \leq |x| + |y|$,  
(7) $|x - y| \leq |x| + |y|$.  

**Axiom of the Rational Numbers** This is actually a theorem but we can use this property of the rational numbers, denoted $\mathbb{Q}$. For any $r, s \in \mathbb{R}$ satisfying $r < s$, there exists $q \in \mathbb{Q}$ such that $r < q < s$. This property is known as $\mathbb{Q}$ is dense in $\mathbb{R}$.  

### 3. Metrics

**Definition 3.1.** Let $X$ be a set. A **metric** (or distance function) on $X$ is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all elements $x, y, z \in X$:  

i. $d(x, y) \geq 0$;  
ii. $d(x, y) = 0$ if and only if $x = y$;  
iii. $d(x, y) = d(y, x)$;  
iv. $d(x, y) \leq d(x, z) + d(z, y)$.  

A **metric space** is a pair $(X, d)$ where $d$ is a metric on $X$.  

**Example 3.2** (A. Sharp, R. Krogman). Define on $\mathbb{R}$ $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the pair $(\mathbb{R}, d)$ is a metric space. This metric is known as the **usual metric** on $\mathbb{R}$.  

**Proposition 3.3** (A. Stein). Prove that on $\mathbb{R}^2$ the function defined by  

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$  

makes $(\mathbb{R}^2, d)$ into a metric space. This metric is known as the **usual metric** on $\mathbb{R}^2$. It is also called the Euclidean metric on the plane.  

**Proposition 3.4** (W. Parker). Prove that on $\mathbb{R}^2$ the function defined by  

$$e((x_1, y_1), (x_2, y_2)) = \text{lub}(|x_1 - x_2|, |y_1 - y_2|)$$  

makes $(\mathbb{R}^2, e)$ into a metric space.  

**Proposition 3.5** (C. Sanders). Prove that on $\mathbb{R}^2$ the function defined by  

$$f((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$  

makes $(\mathbb{R}^2, f)$ into a metric space. This metric is known as the **taxi-cab metric** on the plane.  

**Proposition 3.6** (C. Jinghree). Let $(X, d)$ be a metric space. Prove that the function $d' : X \times X \to \mathbb{R}$ defined by  

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$  

is a metric.  

**Proposition 3.7** (C. Corbett). Let $X$ be any set. Consider the function $d : X \times X \to \mathbb{R}$ defined by  

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$  

The pair $(X, d)$ is a metric space. This is the **discrete metric** on $X$.  

Definition 3.8. Let \((X, d)\) be a metric space and \(A, B \subseteq X\) be non-empty subsets of \(X\). We define the distance between \(A\) and \(B\) as the greatest lower bound of the following set

\[
d(A, B) = \text{g.l.b.}\{t \in \mathbb{R} : \exists a \in A, \exists b \in B \text{ such that } t = d(a, b)\}.
\]

When one of the sets is a singleton set, say \(A = \{a\}\) we instead write \(d(a, B)\) and call this the distance from \(a\) to \(B\).

Proposition 3.9. Let \((X, d)\) be a metric space and let \(A, B\) be non-empty subsets of \(X\). For any \(z \in X\) we have the inequality

\[
d(A, B) \leq d(A, z) + d(z, B).
\]

Proposition 3.10. Let \((X, d)\) be a metric space. Let \(A, B\) be non-empty subsets of \(X\), and let \(z \in X\).

i. If \(A \cap B \neq \emptyset\), then \(d(A, B) = 0\).

ii. If \(A \subseteq B\), then \(d(z, B) \leq d(z, A)\). Moreover, if \(d(x, A) = 0\), then \(d(x, B) = 0\).

Proposition 3.11 (C. Sanders). This problem takes place in the usual space of real numbers \((\mathbb{R}, d)\). Let \(A = \{\frac{1}{n} : n \in \mathbb{N}\}\) and \(r \in \mathbb{R}\).

i) Compute \(d(0, A)\).

ii) Compute \(d(r, \mathbb{Q})\).

Proposition 3.12 (C. Jinghreee). This problem takes place in the euclidean plane \((\mathbb{R}^2, d)\). Let \(A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y = \frac{1}{2}\}\). Let \(B = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y = -\frac{1}{2}\}\). Compute \(d(A, B)\).

Definition 3.13. Let \((X, d)\) be a metric space and let \(A\) be a subset of \(X\). The closure of \(A\) in \(X\) is denoted by \(\text{cl}_X A\) and is defined to be the set

\[
\text{cl}_X A = \{x \in X : d(x, A) = 0\}.
\]

The closure of the empty set is defined as the empty set. Also, whenever \(A\) satisfies \(A = \text{cl}_X A\) we call the set \(A\) closed.

Proposition 3.14 (A. Sharp). Let \((X, d)\) be a metric space and let \(A, B\) be subsets of \(X\).

i. If \(A = \{a\}\), then \(\text{cl}_X A = A\). I.e. ”points are closed”.

ii. \(A \subseteq \text{cl}_X A\).

iii. If \(A \subseteq B\), then \(\text{cl}_X A \subseteq \text{cl}_X B\).

iv. \(\text{cl}_X (A \cup B) = \text{cl}_X A \cup \text{cl}_X B\).

Question 3.15 (A. Stein). Is it the case that \(\text{cl}_X (A \cap B) = \text{cl}_X A \cap \text{cl}_X B\)?

Proposition 3.16. Let \((X, d)\) be a metric space and let \(A\) be a subset of \(X\). The set \(\text{cl}_X A\) is closed.

Question 3.17. Let \((X, d)\) be a metric space and let \(A\) be a subset of \(X\). Is it true that for all \(x \in X\), \(d(x, A) = d(x, \text{cl}_X A)\)?

Proposition 3.18 (A. Sharp). Let \((X, d)\) be a metric space. The following statements are true.

i. \(\emptyset\) and \(X\) are closed sets.

ii. If \(A\) and \(B\) are closed subsets of \(X\), then \(A \cup B\) is closed.
iii. If \( \{ A_\alpha : \alpha \in I \} \) is a collection of closed subsets of \( X \), then \( \bigcap_{\alpha \in I} A_\alpha \) is closed.

**Definition 3.19.** Let \((X, d)\) be a metric space. Let \( y \in X \) and \( r > 0 \). We set 
\[ N_r(y) = \{ x \in X : d(x, y) < r \}. \]
This set is called the open disk centered at \( y \) of radius \( r \).

**Proposition 3.20** (R. Krogman).  
(1) Determine \( N_1(0) \) in Proposition 3.3.  
(2) Determine \( N_1(0) \) in Proposition 3.4.  
(3) Determine \( N_1(0) \) in Proposition 3.5.

**Definition 3.21.** Let \((X, d)\) be a metric space and \( B \) a subset of \( X \). We say \( B \) is an open set if for every \( y \in B \) there is an \( r > 0 \) with \( N_r(y) \subseteq B \).

**Proposition 3.22** (R. Rohan). Let \((X, d)\) be a metric space. A subset \( B \) of \( X \) is open if and only if \( X \setminus B \) is closed.

**Proposition 3.23** (R. Krogman). Let \((X, d)\) be a metric space. Let \( y \in X \) and \( r > 0 \). Prove that \( N_r(y) \) is an open set.

**Proposition 3.24.** Let \((X, d)\) be a metric space. Let \( y \in X \) and \( r > 0 \). Prove that the set 
\[ \{ x \in X : d(x, y) \leq r \} \]
is closed. This is called the closed disk of radius \( r \) and we denote it by \( \overline{N_r(y)} \).

**Question 3.25.** Is it the case that \( \text{cl}_X N_r(y) = \overline{N_r(y)} \)?

**Proposition 3.26** (W. McGovern).  
i. \( \emptyset \) and \( X \) are open sets.  
ii. If \( A \) and \( B \) are open subsets of \( X \), then \( A \cap B \) is open.  
iii. If \( \{ A_i : i \in I \} \) is a collection of open subsets of \( X \), then \( \bigcup_{i \in I} A_i \) is open.

**Proposition 3.27** (W. McGovern). Let \( X = \mathbb{Q} \) with the usual metric. What can you say about the set \( N_{\sqrt{2}}(0) \) besides that it is open? It is a clopen subset of \( X \).

**Definition 3.28.** Any function with domain in \( \mathbb{N} \) is called a sequence. A useful way of looking at a sequence \( f : \mathbb{N} \to X \) is as an indexed set \( \{ a_i \}_{i \in \mathbb{N}} \) where formally this means that \( f(i) = a_i \). If we need to specify the co-domain then we can call it an \( X \)-sequence or say a sequence in \( X \).

For a a metric space \((X, d)\) and \( x \in X \), we say the \( X \)-sequence \( \{ a_i \}_{i \in \mathbb{N}} \) converges to \( x \) if it satisfies the following condition.

For every \( \epsilon > 0 \) there is a natural \( N \in \mathbb{N} \) such that for all \( n \geq N \), 
\[ d(x, a_n) < \epsilon. \]

We often write \( a_n \to x \) to denote that the sequence \( \{ a_n \} \) converges to \( x \).

**Proposition 3.29.** Let \((X, d)\) be a metric space, \( x \in X \), and \( A \subseteq X \). Prove that \( x \in \text{cl}_X A \) if and only if there is a sequence in \( A \), say \( \{ a_n \} \), such that \( a_n \to x \).