MAA 4200 Honors Modern Analysis
Fall 2014

1. Sets, Relations, and Functions

To be successful in this course you should, by now, have a strong grasp of logic as well as a good understanding of the model of Set Theory. Recall this is the first order logic whose objects are sets and which has two binary predicates: $=$, $\in$. The following symbols denote the standard boolean operators of disjunction, conjunction, conditional, negation, and bi-conditional: $\vee, \wedge, \rightarrow, \neg, \leftrightarrow$. Our main symbols for quantifiers are: $\forall$ (universal) and $\exists$ (existential).

The sentence $x \in A$ means that x is a member of A. We will write $x \notin A$ instead of $\neg(x \in A)$.

Definition 1.1. Suppose A and B are sets. We write $A \subseteq B$ (and say A *is a subset of* B) if every element of A is an element of B. This means that

$$\forall x(x \in A \rightarrow x \in B).$$

We now recall our axioms of Set Theory.

Axiom of Extensionality Two sets A and B are equal if and only if they have the same elements, i.e. $A \subseteq B$ and $B \subseteq A$.

Axiom of Set Existence There exists a set which does not contain any elements. This set is denoted by $\emptyset$ and called the *empty set*.

Axiom of Specification To every set A and to every well-defined formula $S(x)$ there corresponds a set B whose elements are exactly those elements $x \in A$ for which $S(x)$ holds. This set is denoted by $\{x \in A : S(x)\}$.

Axiom of Unions We will assume that every collection of sets (which contains at least one set) can be written as $\{A_i\}_{i \in I}$ where I denotes some index set. The axiom of union says that there is a set, denoted $\bigcup_{i \in I} A_i$ which consists of precisely those elements that belong to at least one of the A_i. When the index set is finite, say $i_1, \cdots, i_n$ we instead write $A_{i_1} \cup \cdots \cup A_{i_n}$.

Proposition 1.2. *For any sets A, B, C*

1. $A \cup B = B \cup A$,
2. $A \cup (B \cup C) = (A \cup B) \cup C$,
3. $A \cup A = A$,
4. $A \subseteq B$ *if and only if* $A \cup B = B$.

Lemma 1.3. *Let $\{A_i\}_{i \in I}$ be a family of sets. There is a set whose elements are precisely the elements that belong to A_i for all $i \in I$.*

Remark 1.4. The set given by Lemma 1.3 is called the *intersection* of the sets and is denoted by $\bigcap_{i \in I} A_i$. When I is a finite set, say $i_1, \cdots, i_n$ then we instead write $A_{i_1} \cap \cdots \cap A_{i_n}$. If two sets, say A and B satisfy $A \cap B = \emptyset$, then A and B are said to be *disjoint sets*.

Axiom of Power Sets For any set A there exists a set whose elements are precisely the subsets of A. This set is called the *power set of* A and is denoted by $\mathcal{P}(A)$.

Axiom of Ordered Pairs Let A, B be sets. The *cartesian Product of* A *and* B is the set of ordered pairs of the form (a, b) for $a \in A$, $b \in B$; the set is denoted by $A \times B$. When $A = B$ we instead write A^2 (instead of $A \times A$).

Ordered pairs satisfy the property that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

2. Real Numbers

We let $\mathbb{R}$ denote the set of real numbers. The set of positive real numbers is denoted by $\mathbb{R}^+$ and is the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

Definition 2.1. For a set $S \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, we call u an *upper bound of* S if $u \geq x$ for every $x \in S$. Not every subset of $\mathbb{R}$ possesses an upper bound. If there is an upper bound for S, then we say that S is *bounded above*. A *lower bound* and a *bounded below* set are defined analogously.

Let $S \subset \mathbb{R}$ and let $b \in \mathbb{R}$. We say that b is a *least upper bound for* S if it satisfies the following two properties:

i. b is an upper bound of S,
ii. For any other upper bound of S, say $u \in \mathbb{R}$, $b \leq u$.

A greatest lower bound for a set is defined analogously.

Lemma 2.2. *Suppose $S \subseteq \mathbb{R}$. If S has a least upper bound, then it is unique. Similarly, for greatest lower bound. (So we can use the definite article* the *instead of the indefinite article* a.*) We denote the least upper bound (resp. greatest lower bound) if S, by lub(S) (resp. glb(S)).*

Question 2.3. *Is the empty set bounded above (or below)? If so, is there a lub (or glb)?*

Axiom of Real Numbers We assume that the set of real numbers is *Dedekind Complete*, that is, whenever a set is bounded above (resp. below), then the set in question possesses a least upper bound (resp. greatest lower bound). This statement is actually a theorem however we shall simply assume that $\mathbb{R}$ is Dedekind Complete. [Named after Richard Dedekind c. 1831-1916 who developed the foundational theory of the real numbers.]

Proposition 2.4. *Let $\emptyset \neq B \subseteq A \subseteq \mathbb{R}$. If A is bounded above, then so is B and lub(B)$\leq$ lub(A). Analogously for glb.*

Definition 2.5. Consider the following function defined on $\mathbb{R}$: for $x \in \mathbb{R}$

$$|x| = \begin{cases} x & x \in \mathbb{R}^+ \\ -x & x \in \mathbb{R} \smallsetminus \mathbb{R}^+ \end{cases}$$

Proposition 2.6 (R. Rohan)**.** *The absolute value function has the following properties for all $x, y \in \mathbb{R}$:*

(1) $0 \leq |x|$,
(2) $0 = |x|$ *if and only if* $x = 0$,
(3) $|x| = |-x|$,
(4) $x \leq |x|$,

(5) $|xy| = |x||y|$,
(6) $|x + y| \leq |x| + |y|$,
(7) $|x - y| \leq |x| + |y|$,

Axiom of the Rational Numbers This is actually a theorem but we can use this property of the rational numbers, denoted $\mathbb{Q}$. For any $r, s \in \mathbb{R}$ satisfying $r < s$, there exists $q \in \mathbb{Q}$ such that $r < q < s$. This property is known as $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

3. Metrics

Definition 3.1. Let X be a set. A *metric* (or distance function) on X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all elements $x, y, z \in X$:

i. $d(x, y) \geq 0$;
ii. $d(x, y) = 0$ if and only if $x = y$;
iii. $d(x, y) = d(y, x)$;
iv. $d(x, y) \leq d(x, z) + d(z, y)$.

A *metric space* is a pair (X, d) where d is a metric on X.

Example 3.2 (A. Sharp, R. Krogman)**.** Define on $\mathbb{R}$ $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the pair $(\mathbb{R}, d)$ is a metric space. This metric is known as the *usual metric on* $\mathbb{R}$.

Proposition 3.3 (A. Stein)**.** *Prove that on* $\mathbb{R}^2$ *the function defined by*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

makes $(\mathbb{R}^2, d)$ *into a metric space. This metric is known as the usual metric on* $\mathbb{R}^2$*. It is also called the Euclidean metric on the plane.*

Proposition 3.4 (W. Parker)**.** *Prove that on* $\mathbb{R}^2$ *the function defined by*

$$e((x_1, y_1), (x_2, y_2)) = lub(|x_1 - x_2|, |y_1 - y_2|)$$

makes $(\mathbb{R}^2, e)$ *into a metric space.*

Proposition 3.5 (C. Sanders)**.** *Prove that on* $\mathbb{R}^2$ *the function defined by*

$$f((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

makes $(\mathbb{R}^2, f)$ *into a metric space. This metric is known as the taxi-cab metric on the plane.*

Proposition 3.6 (C. Jinghree)**.** *Let* (X, d) *be a metric space. Prove that the function* $d' : X \times X \to \mathbb{R}$ *defined by*

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a metric.

Proposition 3.7 (C. Corbett)**.** *Let* X *be any set. Consider the function* $d : X \times X \to \mathbb{R}$ *defined by*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

The pair (X, d) *is a metric space. This is the discrete metric on* X.

Definition 3.8. Let (X, d) be a metric space and $A, B \subseteq X$ be non-empty subsets of X. We define the *distance between* A and B as the greatest lower bound of the following set

$$d(A, B) = g.l.b.\{t \in \mathbb{R} : \exists a \in A, \exists b \in B \text{ such that } t = d(a, b)\}.$$

When one of the sets is a singleton set, say $A = \{a\}$ we instead write $d(a, B)$ and call this the *distance from* a *to* B.

Proposition 3.9. *Let (X, d) be a metric space and let A, B be non-empty subsets of X. For any $z \in X$ we have the inequality*

$$d(A, B) \leq d(A, z) + d(z, B).$$

Proposition 3.10. *Let (X, d) be a metric space. Let A, B be non-empty subsets of X, and let $z \in X$.*
i. If $A \cap B \neq \emptyset$, then $d(A, B) = 0$.
ii. If $A \subseteq B$, then $d(z, B) \leq d(z, A)$. Moreover, if $d(x, A) = 0$, then $d(x, B) = 0$.

Proposition 3.11 (C. Sanders)**.** *This problem takes place in the usual space of real numbers $(\mathbb{R}, d)$. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $r \in \mathbb{R}$.*
i) Compute $d(0, A)$.
ii) Compute $d(r, \mathbb{Q})$.

Proposition 3.12 (C. Jinghree)**.** *This problem takes place in the euclidean plane $(\mathbb{R}^2, d)$. Let $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0$ and $y = \frac{1}{x}\}$. Let $B = \{(x, y) \in \mathbb{R}^2 : x \geq 0$ and $y = -\frac{1}{x}\}$. Compute $d(A, B)$.*

Definition 3.13. Let (X, d) be a metric space and let A be a subset of X. The *closure of A in X* is denoted by $cl_X A$ and is defined to be the set

$$cl_X A = \{x \in X : d(x, A) = 0\}.$$

The closure of the empty set is defined as the empty set. Also, whenever A satisfies $A = cl_X A$ we call the set A *closed*.

Proposition 3.14 (A. Sharp)**.** *Let (X, d) be a metric space and let A, B be subsets of X.*

i. *If $A = \{a\}$, then $cl_X A = A$. I.e. "points are closed".*
ii. *$A \subseteq cl_X A$.*
iii. *If $A \subseteq B$, then $cl_X A \subseteq cl_X B$.*
iv. *$cl_X(A \cup B) = cl_X A \cup cl_X B$.*

Question 3.15 (A. Stein)**.** *Is it the case that $cl_X(A \cap B) = cl_X A \cap cl_X B$?*

Proposition 3.16. *Let (X, d) be a metric space and let A be a subset of X. The set $cl_X A$ is closed.*

Question 3.17. *Let (X, d) be a metric space and let A be a subset of X. Is it true that for all $x \in X$, $d(x, A) = d(x, cl_X A)$?*

Proposition 3.18 (A. Sharp)**.** *Let (X, d) be a metric space. The following statements are true.*

i. *$\emptyset$ and X are closed sets.*
ii. *If A and B are closed subsets of X, then $A \cup B$ is closed.*

iii. *If $\{A_\alpha : \alpha \in I\}$ is a collection of closed subsets of X , then $\cap_{\alpha \in I} A_\alpha$ is closed.*

Definition 3.19. Let (X, d) be a metric space. Let $y \in X$ and $r > 0$. We set

$$N_r(y) = \{x \in X : d(x, y) < r\}.$$

This set is called the *open disk centered at y of radius r.*

Proposition 3.20 (R. Krogman)**.** (1) *Determine $N_1(0)$ in Proposition 3.3.*
(2) *Determine $N_1(0)$ in Proposition 3.4.*
(3) *Determine $N_1(0)$ in Proposition 3.5.*

Definition 3.21. Let (X, d) be a metric space and B a subset of X. We say B is an *open set* if for every $y \in B$ there is an $r > 0$ with $N_r(y) \subseteq B$

Proposition 3.22 (R. Rohan)**.** *Let (X, d) be a metric space. A subset B of X is open if and only if $X \setminus B$ is closed.*

Proposition 3.23 (R. Krogman)**.** *Let (X, d) be a metric space. Let $y \in X$ and $r > 0$. Prove that $N_r(y)$ is an open set.*

Proposition 3.24. *Let (X, d) be a metric space. Let $y \in X$ and $r > 0$. Prove that the set*

$$\{x \in X : d(x, y) \le r\}$$

is closed. This is called the closed disk of radius r and we denote it by $\overline{N_r(y)}$.

Question 3.25. *Is it the case that $cl_X N_r(y) = \overline{N_r(y)}$?*

Proposition 3.26 (W. McGovern)**.** i. *$\emptyset$ and X are open sets.*
ii. *If A and B are open subsets of X, then $A \cap B$ is open.*
iii. *If $\{A_i : i \in I\}$ is a collection of open subsets of X, then $\cup_{i \in I} A_i$ is open.*

Proposition 3.27 (W. McGovern)**.** *Let $X = \mathbb{Q}$ with the usual metric. What can you say about the set $N_{\sqrt{2}}(0)$ besides that it is open?* It is a clopen subset of X.

Definition 3.28. Any function with domain in $\mathbb{N}$ is called a *sequence*. A useful way of looking at a sequence $f : \mathbb{N} \to X$ is as an indexed set $\{a_i\}_{i \in \mathbb{N}}$ where formally this means that $f(i) = a_i$. If we need to specify the co-domain then we can call it an *X-sequence* or say a *sequence in X*.

For a a metric space (X, d) and $x \in X$, we say the X-sequence $\{a_i\}_{i \in \mathbb{N}}$ *converges* to x if it satisfies the following condition.

For every $\epsilon > 0$ there is a natural $N \in \mathbb{N}$ such that for all $n \ge N$, $d(x, a_n) < \epsilon$.

We often write $a_n \to x$ to denote that the sequence $\{a_n\}$ converges to x.

Proposition 3.29. *Let (X, d) be a metric space, $x \in X$, and $A \subseteq X$. Prove that $x \in cl_X A$ if and only if there is a sequence in A, say $\{a_n\}$, such that $a_n \to x$.*