Exercise 0.1. *Prove that $\mathbb{N}$ is an inductive set.*

Proof. By Axiom we know there exists an inductive set, call it J. Then the set of naturals is defined as

$$\mathbb{N} = \{x \in J : \forall y(y \text{ Inductive } \to x \in y)\}.$$

It follows that $\mathbb{N}$ is the set of all elements which belong to every inductive set. In order to show that $\mathbb{N}$ is inductive we need demonstrate two things: 1) $0 \in \mathbb{N}$, 2) if $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$.

For the first property if I is any inductive set, then by definition of inductive set, $0 \in I$. Since I is an arbitrary inductive set it follows that 0 belongs to every inductive set. In other words, $0 \in \mathbb{N}$. For the second property, suppose $n \in \mathbb{N}$. This means that n belongs to every inductive set. Thus, for any inductive set, say I, we have $n \in \mathbb{N}$. By definition if inductive set $n^+ \in I$. Since I is an arbitrary inductive set it follows that n^+ belongs to every inductive set, i.e.$n^+ \in \mathbb{N}$. □

Exercise 0.2. *Prove that the following sets are equal.*

(1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(2) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(3) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Proof. 1. Let $(a, x) \in A \times (B \cup C)$. This means that $a \in A$ and $x \in B \cup C$. It logically follows by the distributive laws that $a \in A$ and $x \in B$, or $a \in A$ and $x \in C$. This means that $(a, x) \in (A \times B) \cup (A \times C)$. Thus every element of $A \times (B \cup C)$ is an element of $(A \times B) \cup (A \times C)$. Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

For the reverse containment let $(a, x) \in (A \times B) \cup (A \times C)$. This means that either $a \in A$ and $x \in B$, or $a \in A$ and $x \in C$. It follows that $a \in A$, and either $x \in B$ or $x \in C$. Thus, $(a, x) \in A \times (B \cup C)$. So $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Since both containments hold it follows that the sets in question are equal.

The proof of the other ones are similar.

□

Exercise 0.3. *Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Prove that*

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

Proof. Recall that given two relations, say X from I to J and Y from J to K the composition is defined as

$$Y \circ X = \{(x, y) \in I \times K | \exists z \in J[(x, j) \in X \wedge (j, y) \in Y\}.$$

Therefore,

$$\begin{aligned} T \circ (S \circ R) &= \{(a, d) \in A \times D | \exists c \in C[(a, c) \in S \circ R \wedge (c, d) \in T]\} \\ &= \{(a, d) \in A \times D | \exists c \in C \exists b \in B[(a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T]\} \\ &= \{(a, d) \in A \times D | \exists b \in B \exists c \in C[(a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T]\} \\ &= \{(a, d) \in A \times D | \exists b \in B[(a, b) \in R \wedge (b, d) \in T \circ S]\} \\ &= (T \circ S) \circ R \end{aligned}$$

□

Exercise 0.4. *Suppose $f : A \to B$ and $g : B \to C$ are functions. Set $h = g \circ f$.*

(1) *Prove that if f, g are injective, then h is injective.*
(2) *Prove that if h is surjective, then so is g.*
(3) *Prove that if h is injective and f is surjective, then g is injective.*

Proof. 1. In order to show that h is injective we suppose that $x, y \in A$ and that $h(x) = h(y)$. We need to demonstrate that $x = y$. By definition of composition the first and third equalities in

$$g(f(x)) = h(x) = h(y) = g(f(y))$$

hold. Since we are assuming that g is injective it follows that $f(x) = f(y)$. But we are also assuming that f is injective. Therefore, $x = y$. We conclude that h is injective.

2. To show that $g : B \to C$ is surjective, let $c \in C$. Since h is surjective there is some $a \in A$ such that $h(a) = c$. Set $b = f(a) \in B$. Then

$$g(b) = g(f(a)) = (g \circ f)(a) = h(a) = c.$$

Since $c \in C$ was arbitrarily chosen it follows that for each $c \in C$ there is a $b \in B$ such that $g(b) = c$. It follows that g is surjective.

3. Suppose h is injective and f is surjective. We would like to prove that g is injective. To that end suppose that $b_1, b_2 \in B$ and $g(b_1) = g(b_2)$. Since f is surjective there are $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then

$$h(a_1) = (g \circ f)(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2)) = (g \circ f)(a_2) = h(a_2).$$

Thus it follows that $h(a_1) = h(a_2)$ and since h is assumed to be injective it follows that $a_1 = a_2$. Finally, it follows that by substitution

$$b_1 = f(a_1) = f(a_2) = b_2.$$

Thus, $b_1 = b_2$ from which we conclude that g is injective. □