Conference on Ordered Algebraic Structures

Key Note Speakers: Anthony W. Hager and Richard N. Ball

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The pointfree Daniell integral

Richard N. Ball*, University of Denver

Joint with Aleš Pultr, Charles University, Czech Republic

Abstract

A satisfactory theory of abstract integration, i.e., the study of positive continuous linear functionals on archimedean vector lattices, requires two things. The first is agreement on a unit of measurement; a weak unit is more than is necessary but a truncation is well suited to the task. The second is a satisfactory notion of pointwise convergence on the underlying vector lattice. Surprising as it may seem, the most satisfactory handle on pointwise convergence emerges from pointfree topology.

The classical Daniell approach to integration is to start with an integral on a small family of "simple" functions, and then extend it to a larger family of functions using pointwise convergence. Although it requires transfinite iteration, this can be done just as well in the much more general context of pointfree pointwise convergence. The resulting extension is maximal, both with respect to having the smaller family pointwise dense in the larger, and with respect to having the smaller family epically embedded (in the category of truncated archimedean ℓ-groups) in the larger.

$\mathcal{P}$-essential reflections and hull classes

Ricardo Carrera*, Nova Southeastern

Abstract

$\mathfrak{KNJ}$ (resp., $\mathfrak{KNJS}$) is the category of compact, normal joinfit frames with frame (resp., skeletal frame) homomorphisms. $\mathfrak{KReg}$ (resp., $\mathfrak{KRegS}$) is the category of compact, regular frames with frame (resp., skeletal frame) homomorphisms. We utilize the regular monocoreflection ρ to investigate the morphisms (objects) of $\mathfrak{KNJ}$ and $\mathfrak{KReg}$, introduce the concept of a $\mathfrak{KNJ}$ hull class, and investigate the relationship between the $\mathfrak{KReg}$ and $\mathfrak{KNJ}$ hull classes. In addition, we introduce the concept of a $\mathcal{P}$-essential reflection on $\mathfrak{KNJ}$ and exhibit a correspondence between certain reflective subcategories of $\mathfrak{KRegS}$ and $\mathfrak{KNJS}$. Lastly, we apply our results to extend the concept of a polar function (functorial polar function) to the category $\mathfrak{KNJ}$.

Localic remainders and the Lindelöf property

Themba Dube*, The University of South Africa

Abstract

Let L be a locale. Denote the coframe of its sublocales by $\mathcal{S}(L)$, and the closed sublocale associated with $a \in L$ by $\mathfrak{c}(a)$. A locale is called T_1-*spatial* if it is isomorphic to $\Omega(X)$, for some T_1-space X. In [3], Picado, Pultr and Tozzi prove the following

Proposition. Let L be a T_1-spatial locale. If $S \in \mathcal{S}(L)$ is a join of closed sublocales, and T is any sublocale of L, then

$$S \setminus T = \bigvee \{\mathfrak{c}(x) \mid x \in \mathrm{Max}(L),\, x \in S,\, x \notin T\}.$$

Using this result, I will show how the "remainders" $\beta L \smallsetminus L$ and $\beta L \smallsetminus \lambda L$ can be described in terms of elements (rather than sublocales) of βL. Here, λL designates the Lindelöf reflection of L. I will then give characterizations (some in terms of the ring $\mathrm{C}(L)$ of continuous real-valued functions on L) of when $\beta L \smallsetminus L$ and $\beta L \smallsetminus \lambda L$ are dense sublocales in βL.

I will show that $\beta L \smallsetminus L$ is Lindelöf if and only if L is of countable type, where the latter is defined for locales exactly as for spaces, subject to replacing subspaces with sublocales. This result extends a 1957 theorem of Henriksen and Isbell [2]. The talk is based on [1].

References

[1] T. Dube, *Characterizing realcompact locales via remainders.* Georgian Math. J. (to appear).

[2] M. Henriksen and J. R. Isbell, *Some properties of compactifications.* Duke Math. J. **25** (1957), 83–105.

[3] J. Picado, A. Pultr and A. Tozzi, *Joins of closed sublocales.* Houst. J. Math. (to appear).

Internal Neighbourhood Spaces

Partha Pratim Ghosh*, The University of South Africa

Abstract

The paper generalises the construction of pretopological spaces to a context where the ground category of sets is replaced with an arbitrary finitely complete category equipped with a proper factorisation system and each lattice of *admissible subobjects* is a complete distributive lattice. It is shown that the categories of *internal pretopological/ preneighbourhood spaces* and *internal weak neighbourhood spaces* are topological over the base category. The category of *internal weak neighbourhood spaces* is shown to be bireflective in the category of *internal pretopological spaces*.

The *internal neighbourhood spaces* make a non-full subcategory of the category of *internal weak neighbourhood spaces* and is topological over the non-full subcategory of the base category consisting of all morphisms for which the preimage homomorphism preserve arbitrary joins.

If the base category is taken to be the familiar one of sets and functions, then its *internal neighbourhood spaces* are precisely the topological spaces. However, one could define the full subcategory of *internal neighbourhood spaces* consisting of precisely those objects for which the open neighbourhoods make a frame. In such a case, this category of *internal topological spaces* is topological over the same non-full subcategory of the base category alluded to previously, if and only if, it is reflective inside *internal neighbourhood spaces*, if and only if, each object has a largest topology!

Topologicity provides a good description of limits whichever exists in the base category. The regular epimorphisms and the hereditary regular epimorphisms of *internal pretopological spaces* and *internal weak neighbourhood spaces* are completely described.

The work on this project is ongoing and it still remains to resolve a proper definition for the analogue of pseudotopological spaces.

Adjoining a strong unit to an archimedean ℓ-group

Anthony W. Hager*, Wesleyan University

Joint with Philip Scowcroft, Wesleyan University

Abstract

All ℓ-groups G,H, are archimedean. The class $\mathbf{W}^*$ consists of the H with a strong unit, and $\mathbf{SW}^*$ consists of the G which embed into an H in $\mathbf{W}^*$.

(A) Theorem. $C(X)$ is in $\mathbf{SW}^*$ iff X is pseudocompact. (X is a Tychonoff space, or a completely regular frame.)

This looks easy, but devolves from the result for $C(N)$, which depends on a rather tricky lemma of Conrad-Martinez, using "basic representations." $C(N)$ is involved in other ways, through its associated uncountable cardinal numbers called p, $\mathfrak{b}$, $\mathfrak{c}$ (in increasing order). We look inside $C(X)$s. For any G: There is the a-order (a for archimedean), and dG (resp. lG) is the minimum size of an a-cofinal (a-initial) subset of G^+.

(B) **Theorem.** Suppose G is a sub-ℓ-group of $C(X)$,and X contains densely the union of compact sets X_i ($i \in I$). Then, G is in $\mathbf{SW}^*$ if either (I is countable and $dG < \mathfrak{b}$) or ($|I| < \mathfrak{b}$ and dG is countable).

(C) **Theorem.** G is in $\mathbf{SW}^*$ if either (dG is countable and $lG < \mathfrak{b}$) or ($dG < p$ and lG is countable). Under [CH] (or weaker), the two theorems in B (resp.,C) are one. Under other axioms, they are not. Examples (sometimes under "axioms") show sharpness of hypotheses.

Concerning the summand intersection property in function rings

Oghenetega Ighedo*, The University of South Africa

Abstract

A ring has the summand intersection property (SIP, for short) if the intersection of any collection of direct summands is a direct summand. The ring of continuous real-valued functions on a completely regular locale L is denoted by $\mathcal{R}L$. We characterize in terms of elements and sublocales the locales L for which $\mathcal{R}L$ has SIP. We show that if L is covered by its connected components, and the connected components are open, then $\mathcal{R}L$ has SIP. These results extend Azarpanah's results in [1] for the rings $C(X)$.

References

[1] F. Azarpanah, *Sum and intersection of summand ideals in* $C(X)$, Comm. Algebra **27** (1999), 5549–5560.

Pseudo-valuation domains and $C(X)$

Lee Klingler*, Florida Atlantic University

Joint with Warren Wm. McGovern, Florida Atlantic University

Abstract

In joint work with Warren McGovern, we consider integral domains formed as factor rings $C(X)/P$, where $C(X)$ is the ring of continuous real-valued functions on a topological space X, and P is a prime ideal of $C(X)$. It is well-known that the set of prime ideals of $C(X)/P$ is totally ordered by inclusion, but that, in general, $C(X)/P$ need not be a valuation domain. We explore several possible conditions (including pseudo-valuation domain) between these two extremes, concentrating on integral domains of the form $C(X)/P$. We prove necessary and sufficient conditions on $C(X)/P$ (in terms of the order inherited from $C(X)$) that it be a pseudo-valuation domain. We also give examples showing that $C(X)/P$ may be a pseudo-valuation domain without being a valuation domain, but it need not be a pseudo-valuation domain.

Embeddings of o-Groups of Finite Archimedean Rank

Ramiro H. Lafuente-Rodriguez*, University of South Dakota

Abstract

The classical results in the theory of lattice ordered groups that every lattice ordered group can be embedded in a divisible lattice ordered group, proved by W. C. Holland, and that not every totally ordered group can be embedded in a divisible totally ordered group, proved by V. Bludov, led to the formulation and study of similar questions in subclasses of totally ordered groups. We formulate this question in the context of totally ordered groups of finite Archimedean rank and prove that for every $n > 2$ there exists an o-group of finite rank n such that G cannot be embedded in a divisible o-group of archimedean rank n, and we also prove that G can be embedded in an o-group of finite archimedean rank greater than n.

Paralocalic groups

James Madden*, Louisiana State University

Joint with Martin Mugochi, University of Namibia

Abstract

A paratopological group is a group with a topology in which multiplication, but not inversion, is continuous. A paralocalic group is a pointfree version of this. The first problem is to create an appropriate definition. To do this, we use the fact that every paratopological group becomes a topological group if endowed with the weakest topology stronger than both the given topology and its inverse. There are (easy) examples of paratopological groups that is also paralocalic. The Sorgenfrey line is a paratopological group that is not paralocalic. The proof involves some interesting basic geometry in the plane.

Clopen π-bases in the spaces of minimal primes and maximal d-subgroups

Warren Wm. McGovern*, Wilkes Honors College, FAU

Abstract

This will be a talk about structure spaces in ℓ-groups. Let G denote an ℓ-group, not necessarily abelian but with a weak order unit. In the past, we have addressed questions of the form when does the inverse topology on the space of minimal prime subgroups of G, denoted $\mathrm{Min}(G)^{-1}$, have certain topological properties, e.g. when is it Hausdorff (iff G is lamron, or when is it zero-dimensional (iff G is weakly complemented). We began with a simple question. When does $\mathrm{Min}(G)^{-1}$ have a clopen π-base? We characterize this case and share how this led us to consider the same question on $\mathrm{Max}_d(G)$, the space of maximal d-subgroups of G. Our foray into the fields have uncovered some interesting theorems, some involving fusible ℓ-groups and others with what we are calling c-subgroups.

We will finish the talk with a closer inspection of archimedean ℓ-groups. A comparison to work by several authors (e.g. Dashielle, Hager, Henriksen, Vermeer, and Woods) will be attempted.

Unique Decomposition of Direct Sums of Ideals

IdealsAkeel Omairi*, Florida Atlantic University

Joint with Lee Klingler, Florida Atlantic University

Abstract

Let R be a commutative Noetherian ring. We say that R has the unique decomposition into ideals (UDI) property if each finite direct sum of ideals of R is uniquely decomposable as a direct sum of indecomposable R-ideals. For an integral domain R, Goeters and Olbering showed that R has UDI if and only if R has at most one nonprincipal maximal ideal and has UDI locally at that nonprincipal maximal ideal (if it exists). For local domain R, they gave necessary and sufficient conditions that R has UDI in terms of its integral closure. Their results were extended to reduced (commutative Noetherian) rings by Ay and Klingler. We show that if R is any commutative Noetherian ring, then R has UDI if and only if R has at most one nonprincipal maximal ideal and has UDI locally at that nonprincipal maximal ideal (if it exists). We also give an example of a ring without UDI but which has UDI modulo its nilradical, so that the UDI property does not lift modulo the nilradical.

ℓ-groups existentially closed in W

Philip Scowcroft*, Wesleyan university

Abstract

If $\mathcal{K}$ is a class of algebras and $A \in \mathcal{K}$, A is existentially closed (e.c.) in $\mathcal{K}$ just in case every finite system of equations and inequations, involving elements of A, is solvable in A if solvable in some $B \in \mathcal{K}$ extending A. Weispfenning found a useful set of axioms for the e.c. Abelian ℓ-groups; but this class of ℓ-groups cannot be axiomatized by first-order statements, and many techniques from model theory do not apply to the e.c. Abelian ℓ-groups. At the last OAL meeting McGovern asked for a description of the ℓ-groups e.c. in W, the class of Archimedean ℓ-groups with distinguished weak order unit. Though neither W nor the class of e.c. algebras in W has first-order axioms, the e.c. algebras in W are the Archimedean ℓ-groups satisfying a set T of first-order axioms with many nice properties. The existence of T suggests that some techniques from model theory may apply to W, and this talk will begin to explore the possibilities.

An e.c ℓ-group via Fraïssé's construction

Brian Wynne*, Lehman College, CUNY

Abstract

I will discuss how a construction of Fraïssé can be used to produce a new example of an existentially closed Abelian lattice-ordered group.