

# Conference on Ordered Algebraic Structures

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**Tightness relative to some (co)reflections in topology**

Richard N. Ball\*, B. Banaschewski, T. Jakl, A. Pultr, and Joanne Walters-Wayland\*

**Abstract**

The problem we address can be formulated as follows. We have a monoreflection of a category  $\mathcal{A}$  onto a subcategory  $\mathcal{B}$ ; the question is whether a given  $B \in \mathcal{B}$  is the reflection of a proper subobject. We start with a well known specific instance of this question, namely the fact that a compact metric space is never the Čech-Stone compactification of a proper subspace. We show that this holds also in the pointfree setting, i.e., we show that a compact regular metrizable locale (frame) is never the Čech-Stone compactification of a proper sublocale. This is a stronger result than the classical one, but not because of any increase of scope; after all, assuming weak choice principles, every compact regular locale is the topology of a compact Hausdorff space. The increased strength derives from the conclusion, for in general every space has many more sublocales than subspaces. We then extend the analysis from metric locales to the broader class of perfectly normal locales, i.e., those whose frame of open sets consists entirely of cozero elements. At the opposite extreme from these results, we show that an extremally disconnected locale is a compactification of each of its dense sublocales.

The result on metrizable frames can be translated into an algebraic statement about principal (completely regular) ideals in frames. Thus our first proof, which depends heavily on topological reasoning, begs the question of an alternative algebraic proof, and we give one. Finally, we analyze the same phenomena, also in the pointfree setting, for the 0-dimensional compact reflection and for the Lindelöfication.

**Multiplicative structure of biorthomorphisms and embedding of orthomorphisms**

Karim Boulabiar\*, W. Brahmi

**Abstract**

Let  $\mathcal{X}$  be an Archimedean vector lattice. A biorthomorphism on  $\mathcal{X}$  is a bilinear map from  $\mathcal{X} \times \mathcal{X}$  into  $\mathcal{X}$  which is an orthomorphism on  $\mathcal{X}$  in each variable separately. The set of such biorthomorphisms is denoted by  $\text{Orth}(\mathcal{X}, \mathcal{X})$ . We prove that if  $\text{Orth}(\mathcal{X}, \mathcal{X})$  is not trivial then  $\text{Orth}(\mathcal{X}, \mathcal{X})$  is equipped with a structure of  $f$ -algebra, giving thus a complete answer to a question asked quite recently by Buskes and Yilmaz. On the other hand, we assume that  $\mathcal{X}$  is a semiprime (i.e., reduced)  $f$ -algebra and we show that if  $\mathcal{X}$  is either Dedekind-complete or uniformly-complete with a weak order unit, then the set of all orthomorphisms on  $\mathcal{X}$  has an order ideal copy in  $\text{Orth}(\mathcal{X}, \mathcal{X})$ . Notice that the Dedekind-complete case has been obtained again by Buskes and Yilmaz in a completely different way. This talk is based upon a joint research with Wael Brahmi.

## Reverse Mathematics of the Grätzer-Schmidt Theorem

Katie Brodhead, et. al.

### Abstract

The Grätzer-Schmidt theorem of lattice theory states that each algebraic lattice is isomorphic to the congruence lattice of an algebra. A lattice is algebraic if it is complete and generated by its compact elements. We study the reverse mathematics of this theorem. We also show that: the set of indices of computable lattices that are complete is  $\Pi_1^1$  complete; the set of indices of computable lattices that are algebraic is  $\Pi_1^1$  complete; the set of compact elements of a computable lattice is always  $\Pi_1^1$  and can be  $\Pi_1^1$ -complete; and the set of compact elements of a distributive computable lattice is always  $\Pi_0^3$  and can be  $\Pi_1^1$ -complete.

This is joint work with Mushfeq Khan (University of Hawaii at Mānoa), Bjrn KjosHanssen (University of Hawaii at Mānoa), William Lampe (Hawaii at Mānoa), Paul Nguyen (University of Hawaii at Mānoa), and Richard Shore (Cornell University).

## “Scrimgerized” Nilpotent and Engel Lattice-ordered Groups

Michael Darnel

### Abstract

For  $n$  a positive integer,  $\mathcal{L}_n$  is the variety of  $\ell$ -groups satisfying the equation  $[x^n, y^n] = e$ . Many of the well known but striking features of the  $\mathcal{L}_n$  varieties carry over to varieties satisfying the laws  $[x^n, y^n, z^n] = e$  and  $[x^n, y^n, y^n] = e$ , such as the structure of finitely generated subdirectly irreducible  $\ell$ -groups and the respective intersections with the representable variety.

Some interesting examples of subdirectly irreducible  $\ell$ -groups that are neither nilpotent nor Scrimger will be shown, with discussion of their varieties.

## Commutative rings in which components of zero are essential

Themba Dube

### Abstract

For a prime ideal  $P$  of a commutative ring  $A$  with identity, we denote (as usual) by  $O_P$  its component of zero, that is, the set of those members of  $P$  which are annihilated by non-members of  $P$ . We say  $A$  is *essentially good* if  $O_P$  is an essential ideal whenever  $P$  is essential. I will present a characterization of these rings in terms of properties of their frames of radical ideals. I will give an outline of the proof that the direct product of any collection of essentially good rings is essentially good. The ring  $C(X)$  is essentially good if and only if the underlying set is infinite. Replacing  $O_P$  with the pure part of  $P$ , we obtain a stronger variant of essential goodness, which is still characterizable in terms of the frame of radical ideals.

## The $\sigma$ -property in $C(X)$

Anthony W. Hager

### Abstract

A vector lattice is a real linear space with a compatible lattice-order. Examples are  $C(X)$  (continuous functions from the topological space  $X$  to the reals), and any abstract "measurable functions mod null functions". The  $\sigma$ -property of a vector lattice  $A$  is (s) For each sequence  $\{a(n)\}$  in  $A^+$ , there are a sequence  $\{p(n)\}$  of positive reals and  $a \in A$  for which  $p(n)a(n) < a$  for each  $n$ .  $C(X)$ , for  $X$  compact (trivial); Lebesgue Measurable functions mod Null (not trivial—connected with Egoroff's Theorem). An application: If a quotient  $A/I$  has (s), then the quotient map lifts disjoint sets to disjoint sets. Here, we consider which  $C(X)$  have (s). For example: For discrete  $X$ ,  $C(X)$  has (s) iff the cardinality of  $X <$  the bounding number  $\mathfrak{b}$ . For metrizable  $X$ ,  $C(X)$  has (s) iff  $X$  is locally compact and each open cover has a subcover of size  $< \mathfrak{b}$ . (This much studied " $\mathfrak{b}$ " is the minimum among cardinals  $m$  for which each family of functions from the positive integers  $\mathbb{N}$  to  $\mathbb{N}$  of size  $m$  is bounded in the order of eventual domination for such functions. It is uncountable, no bigger than  $\mathfrak{c}$ , and regular. In ZFC, not much more can be said.)

## Higher order $z$ -ideals in commutative rings

Oghenetega Ighedo\*, T. Dube

### Abstract

I will present ideals of commutative rings that resemble Mason's [1]  $z$ -ideals. For each positive integer  $n$ , we say an ideal of a commutative ring  $A$  is a  $z^n$ -ideal in case it has the property that if  $a, b \in A$  belong to the same maximal ideals of  $A$ , and  $a^n \in I$ , then also  $b^n \in I$ . The set of all  $z^n$ -ideals of  $A$  is denoted by  $\mathfrak{Z}^n(A)$ . This gives an ascending chain

$$\mathfrak{Z}(A) \subseteq \mathfrak{Z}^2(A) \subseteq \mathfrak{Z}^3(A) \subseteq \dots$$

of collections of ideals, starting with the collection of  $z$ -ideals. I will give examples of when the chain becomes stationary, and when it ascends without stop, with each collection properly contained in its successor. If time allows, I will show that the assignment  $A \mapsto \mathfrak{Z}^n(A)$  is the object part of a functor  $\mathbf{Rng}_3^{\text{op}} \rightarrow \mathbf{Set}$ , where  $\mathbf{Rng}_3$  denotes the category of commutative rings and homomorphisms that contract  $z$ -ideals to  $z$ -ideals. When the objects are restricted to rings with zero Jacobson radical, the restricted functor reflects epimorphisms, but not monomorphisms.

## References

- [1] G. Mason,  *$z$ -Ideals and prime ideals*, J. Algebra **26** (1973), 280-297.

## Complex algebras of tree-semilattices

Peter Jipsen

### Abstract

The complex algebra of a semilattice is the powerset algebra of the semilattice with union, intersection, complementation and the semilattice operation lifted to subsets. This construction gives a Boolean algebra with a binary operator that is commutative, associative and square-increasing. The class BSL of Boolean semilattices is this variety of BAOs. The representable Boolean semilattices form the subvariety RBSL that is generated by all complex algebras of semilattices. We provide several new quasiequations that hold in all complex algebras of semilattices but not in BSL.

We also find an identity that holds in all tree-semilattices (i.e. the partial order is a tree) and we prove that all finite Boolean semilattices that satisfy this identity and another simple condition are embeddable in complex algebras of finite tree-semilattices.

## Conditions on normal spaces of finite rank that guarantee $C(X)$ is SV

Suzanne Larson

### Abstract

An  $f$ -ring  $A$  is an SV  $f$ -ring if for every minimal prime  $\ell$ -ideal  $P$  of  $A$ ,  $A/P$  is a valuation domain. In an  $f$ -ring, the *rank* of a maximal ideal  $M$ , is the number of minimal prime ideals contained in  $M$  if the set of all such minimal prime ideals is finite, and the rank of  $M$  is infinite otherwise. The rank of an  $f$ -ring  $A$  is the supremum of the ranks of the maximal  $\ell$ -ideals of  $A$ . Although in general, neither being an SV  $f$ -ring nor an  $f$ -ring of finite rank implies the other, the rank of maximal ideals plays a large role in the study of SV  $f$ -rings. It is known that for every completely regular topological space  $X$ , if  $C(X)$  is SV, then  $C(X)$  has finite rank. We will show the converse holds for certain classes of normal spaces. For normal spaces for which  $C(X)$  has rank 2, a characterization of SV spaces will be given.

## Frames without top as a domain for representing archimedean $\ell$ -groups

James Madden

### Abstract

In recent talks, I have described a representation theorem for archimedean  $\ell$ -groups without weak unit that uses presheaves. It is possible to describe this representation in an alternate manner using a natural construction that based on weakening the relations that are used in the weak-unit case to define the representation locale. From the geometric perspective, we add a new point to the reals and to the representation locale. This is related to a construction used by Rick Ball in representing  $\ell$ -groups with truncation. The two approaches (via presheaves and via topless frames) are analogous to the two different ways of thinking about projective space in algebraic geometry. The new point is analogous to the "irrelevant ideal" that arises in the Proj construction.

## Projectable hulls through compactifications of minimal spectral spaces.

R. N. Ball, Vincenzo Marra\*, D. McNeill, A. Pedrini

### Abstract

Any Archimedean lattice-ordered group  $G$  with a strong unit induces a  $G$ -indexed zero-dimensional compactification  $wZ_G$  of its space  $Z_G$  of minimal prime ideals. Using Freudenthal's 1936 Spectral Theorem, we obtain a natural embedding of  $G$  into  $C(wZ_G)$ , which in general does not separate the points of  $wZ_G$ . Main result: the inclusion-minimal extension of this representation of  $G$  that separates the points of  $Z_G$  — namely, the sublattice subgroup of  $C(wZ_G)$  generated by the image of  $G$  along with all characteristic functions of those clopen subsets of  $Z_G$  which are determined by elements of  $G$  — is precisely the projectable hull of  $G$ . This reveals a fundamental relationship between projectable hulls and minimal spectra in the strongly unital case. We do not know how to extend our construction to the weakly unital case. These results are related to several others in the literature. If time allows I will discuss some of them. This is joint work with R. N. Ball, D. McNeill, and A. Pedrini.

## Typings in Frames, Old and New

Jorge Martínez

### Abstract

The goal in this presentation is to review several concepts and constructions, which were introduced or explored in papers as long as fourteen years ago. By looking at these again, in view of some recent categorical developments on epimorphisms, it is hoped that fresh eyes will find these *algebraic* ideas in frame theory stimulating.

## Algebraic Properties of Rings of Germs

Warren Wm. McGovern

### Abstract

For a topological space  $X$  and  $p \in X$ , recall that  $M_p = \{f \in C(X) : f(p) = 0\}$  and  $O_p = \{f \in C(X) : f \text{ vanishes on a neighborhood of } p\}$ . We denote the localization of  $C(X)$  at  $M_p$  by  $\mathfrak{G}_p$ , and recall that this ring is known as the *ring of germs* at  $p$ . It is well-known that  $\mathfrak{G}_p$  is isomorphic to  $C(X)/O_p$ . We shall discuss different algebraic properties concerning the ring of germs, and the topological spaces associated with such things.

## The Dedekind-MacNeille Completion of a Regular $\sigma$ -frame is a Frame

M. Andrew Moshier

### Abstract

Following on the investigation of tightness with respect to Lindelöf frames in this workshop (Rick Ball and Joanne Walters), we consider the range of tight sub-locales of a Lindelöf frame. By definition, these are always dense. But they can be bracketed even farther up in the lattice of sub-locales. We show that every tight sub-locale of a Lindelöf frame  $L$  contains (an isomorphic copy of) the Dedekind MacNeille completion of the lattice of cozeros of  $L$ , and that this copy is itself a sub-locale. Hence, the Dedekind-MacNeille completion of any regular  $\sigma$ -frame is a frame. The proof is a bit surprising because regular *sigma*-frames do not obviously satisfy known conditions on distributive lattices that guarantee the Dedekind-MacNeille completion even to be distributive, let alone frame distributive.

## Yosida frames and strong semisimplicity

Daniele Mundici

### Abstract

Working in the multiple context of lattice ordered abelian groups ( $\ell$ -groups), unital  $\ell$ -groups, and MV-algebras [3], we discuss the relationships between Yosida frames, [2] and the Dubuc-Poveda notion of strong semisimplicity, [1]. We give a positive solution to a problem posed in [2, Remark 5.6].

## References

- [1] E. J. Dubuc, Y. A. Poveda, Representation theory of MV-algebras, *Annals of Pure and Applied Logic* 161 (2010) 1024-1046
- [2] J. Martínez, E. R. Zenk, Yosida frames, *Journal of Pure and Applied Algebra* 204 (2006) 473–492.
- [3] D.Mundici, *Advanced Łukasiewicz calculus and MV-algebras*, Trends in Logic, Vol. 35, Springer-Verlag, Berlin, NY, 2011.



## On convergence in frames

Papiya Bhattacharjee, Themba Dube, Inderasan Naidoo\*

### Abstract

We revisit the cover notion of convergence and clustering in frames that was introduced by Hong [7]. We particularly concern ourselves with general filters formalized in Banaschewski [1] and convergence thereof in the subsequent joint papers of Banaschewski and Hong [2, 3, 4] with emphasis on characterizing the notion of clustering of general filters. We will also look at the notion of a balanced general filter classically presented in Dube [5]. We provide alternate characterizations of clustering of filters via covers, expand on the notion of strong clustering of filters that was introduced by Dube and Naidoo [6] and also consider convergence in algebraic frames, particularly M-frames. On the latter we provide an ultrafilter criterion for the compactness of such regular frames.

## References

- [1] Banaschewski, B., Completion in Pointfree Topology, Lecture Notes in Math. and Appl. Math., Univ. of Cape. Town, SoCat94, No2/(1996).
- [2] Banaschewski, B. and Hong S.S., *Filters and strict extensions of frames*, Kyungpook Math. J. **39** (1999), 215–230.
- [3] Banaschewski, and Hong S.S., *Extension by continuity in pointfree topology*, Appl. Categ. Struct. **8** (2000) 475–486.
- [4] Banaschewski, B. and Hong S.S., *General filters and strict extensions in pointfree topology*, Kyungpook Math. J. **42**(2002) 273–281.
- [5] Dube, T., *Balanced and closed-generated filters in frames*, Quaest. Math. **26** (2003) 73-81.
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- [7] Hong, S.S., *Convergence in frames*, Kyung. Math. J. **35** (1995), 85–91.

## Stone duality above dimension zero

Luca Reggio

### Abstract

It is known since the work of Duskin [2] in 1969 that the category  $\mathbf{KHaus}$  of compact Hausdorff spaces and continuous maps is dually equivalent to a (possibly infinitary) variety of algebras. In 1982 Isbell showed [3] that finitely many finitary operations, along with just one operation of countably infinite arity, suffice to describe the dual of the category  $\mathbf{KHaus}$ . Not only is such an infinitary operation known to be necessary, but Banaschewski and Rosický also proved that the dual of  $\mathbf{KHaus}$  is not axiomatisable by a large class of first-order theories [1, 4].

The problem of providing a tractable equational axiomatisation of the dual category  $\mathbf{KHaus}^{\text{op}}$  has remained open. Using Isbell's insight on the semantic nature of the infinitary operation, and the theory of  $\ell$ -groups and MV-algebras as a key tool, we provide a *finite* axiomatisation. In a precise sense this extends Stone duality from Boolean spaces to compact Hausdorff spaces.

## References

- [1] Bernhard Banaschewski, *More on compact Hausdorff spaces and finitary duality*, *Canad. J. Math.* **36** (1984), no. 6, 1113–1118.
- [2] John Duskin, *Variations on Beck's tripleability criterion*, *Reports of the Midwest Category Seminar, III*, Springer, Berlin, 1969, pp. 74–129.
- [3] John Isbell, *Generating the algebraic theory of  $C(X)$* , *Algebra Universalis* **15** (1982), no. 2, 153–155.
- [4] Jiří Rosický, *Elementary categories*, *Arch. Math. (Basel)* **52** (1989), no. 3, 284–288.

## Infinitely generic Abelian $\ell$ -groups

Philip Scowcroft

### Abstract

If  $\mathcal{A}$  is the class of Abelian  $\ell$ -groups, the class  $\mathcal{I}$  of infinitely generic Abelian  $\ell$ -groups is the unique subclass of  $\mathcal{A}$  with the following properties: every  $M \in \mathcal{A}$  is contained in some  $N \in \mathcal{I}$ ; every inclusion  $N_1 \subseteq N_2$  of elements of  $\mathcal{I}$  is an elementary inclusion  $N_1 \preceq N_2$  (as defined in mathematical logic); and if  $P \in \mathcal{I}$ ,  $Q \in \mathcal{A}$ , and  $Q \preceq P$ , then  $Q \in \mathcal{I}$ . So the relation between  $\mathcal{I}$  and  $\mathcal{A}$  closely resembles the relation between the class of algebraically closed fields and the class of fields, or the relation between the class of real-closed ordered fields and the class of ordered fields. This talk will survey what is known about  $\mathcal{I}$  as well as the mysteries that remain.

## Topological locales

Niels Schwartz

### Abstract

Stone Duality (between bounded distributive lattices and spectral spaces) restricts to a duality between frames and a special class of spectral spaces, which I call *locales*. They can be viewed as a realization of the locales in frame theory as topological spaces. The systematic study of these topological locales has been initiated recently. I report about progress and applications.

## Reconstructing semi-algebraic sets from the first order theory of their rings of continuous semi-algebraic functions.

Marcus Tressl

### Abstract

A semi-algebraic set  $X$  is a subset of the euclidean space that is described by a boolean combination of polynomial inequalities. Like in the case of classical algebraic geometry these sets are studied via rings of functions defined on them. For example the ring  $A$  of real valued continuous functions on  $X$  that have a semi-algebraic graph. It is well known that the ring  $A$  knows everything about  $X$ . In this talk I will show how to reconstruct the set  $X$  (in reality: its semi-algebraic homeomorphism type) already from the list of all first order properties of the ring  $A$ .

## Can associativity force commutativity?

Piotr J. Wojciechowski

### Abstract

This seemingly trivial question stems from a formidable proof of Bernau and Huijsmans that every archimedean almost  $f$ -algebra is commutative - a proof independent on associativity of the algebra. Since every archimedean  $f$ -algebra is automatically associative and commutative, in these classes associativity definitely does not force commutativity. Can associativity force commutativity at all? We will scratch the surface of this problem by giving several positive, several negative and several undecided answers. In particular, we will exhibit a class of  $d$ -algebras, where associativity forces commutativity. Also, the outstanding Birkhoff's problem 157 asking whether associativity is essential in the proof that an  $\ell$ -algebra with positive unity which is a weak order unit is an almost  $f$ -algebra, magnifies its importance in our context. The research in this area is practically endless and touches all classes of rings and abelian groups.