Rationing-Based Price Discrimination *

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Abstract

This paper provides a theory of rationing where rationing functions as an effective mechanism for second degree price discrimination by a monopoly seller. When a seller charges multiple prices on homogenous products to all consumers, supply at the lowest price is limited and rationed among consumers. The supply shortage differentiates products sold at the lowest price and those sold at a higher price. When high-valuation consumers identify themselves at the higher price, the seller may extract more consumer surplus and increase his profit. In the paper, we address two common rationing-based price discrimination strategies, multiple-price menu and premium advance selling. We also show that rationing-based price discrimination can be combined with other classical price discrimination strategies to further increase the seller’s profit.
1. Introduction

In markets where products have homogeneous quality and consumers have unit demands, multiple prices are still frequently observed as sellers adopt price discrimination strategies to maximize their profits. A common phenomenon accompanying the multiple pricing in such markets is rationing. That is, when sellers charge different prices (*high price* and *low price*, as simplified in this paper), supply at the low price is limited and rationed among consumers. For example, on Southwest Airline’s website, a same one-way flight is sold at both “Internet fare” and “fun fare” but with almost identical conditions of purchase and refund. Tickets at “Internet fare” always sell out quickly. Similarly, concert tickets in the same seat section are sold both at the box offices for the face price and online for the face price plus some “convenience fees.” However, consumers who want to buy at the box office often have to wait in a long line. In another example, stores sometimes offer large discounts on certain products with very limited supply. Even “early birds” are not guaranteed to get those deals.

Why does rationing often happen when a seller charges multiple prices? Intuitively, when homogeneous products are sold at both the low price and the high price, supply at the low price is expected to run out quickly, since even those who can afford the high price would like to buy at the low price. However, when some of those consumers who can afford the high price buy at the low price in the rationing, does the seller maximize his profits, especially in a market where demand can be well predicted? Or, could the seller increase his profits by raising the low price a little bit or even by abandoning the multiple pricing scheme altogether?
This paper provides an explanation in which the seller intentionally introduces rationing to support his second degree price discrimination strategy. We present the intuition using a base model where a monopoly seller sells homogeneous products through a two-price (the high price and the low price) menu instead of a uniform monopoly price (hereafter *monopoly price*), and resale is not allowed. The seller satisfies all demand at the high price but rations supply at the low price, introducing a buying frenzy in which only some consumers can secure a product at this price. Consumers who have valuations higher than the high-price but fail to secure a product at the low-price will reveal their valuation by purchasing at the high price. Essentially, the seller price discriminates those consumers from the rest of the consumers.

However, compared with uniform pricing, the rationing-based price discrimination strategy is not always profitable for the seller. The profitability is determined by the seller’s capacity constraint and the shape of the demand curve. Under the rationing strategy, the rationing ratio is crucial to the seller’s profit, as, on one hand, it affects the sale at the high price, and on the other hand, reflects the profit loss at the low price. The seller can adjust the supplies at the two prices to optimize the rationing ratio, but his performance is restrained by his capacity constraint. Meanwhile, the rationing ratio is also determined by the demand at each price level.

In this paper we conduct a comprehensive analysis of the conditions of profitability given the seller’s capacity constraint and the shape of the demand curve. We first show the necessary and sufficient condition on the demand curve, under which rationing-based price discrimination is more profitable than uniform pricing without increasing the aggregate sales quantity. According to this condition, a rationing strategy
is not profitable if the distribution function of consumer valuation has the increasing hazard rate (IHR), which is a stylized assumption in many papers on price discrimination. However, rationing is very likely to be profitable if the distribution function is discrete. This is also a common assumption supporting many other papers on price discrimination. We give sufficient conditions on the continuous distribution function and show a numerical example. Second, we prove that a rationing strategy is no more profitable than uniform pricing if the seller has unlimited capacity. Third, we consider the case when the seller strategically wastes some capacity under uniform pricing: in this case, rationing-based price discrimination is more profitable than uniform pricing.

Besides the two-price menu shown in the base model, rationing-based price discrimination can be realized in other approaches. The first extension model of the paper presents another common rationing strategy — premium advance selling, where a seller sells his products at the high price in the first period, namely, the *advance period*, and then sells at the low price with limited supply in the second period, namely, the *spot period*. Rationing is expected to happen in the spot period and reduces consumers’ expected surplus in the spot period. Consumers with relatively high valuations would rather buy at the high-price in the advance period, since their *ex ante* expected surplus decreases faster than that of consumers with relatively low valuations. In the extension model, because of the timing setup, rationing explicitly functions as a vehicle supporting the self-selection mechanism. However, as consumers with relatively high valuations don’t always join in the rationing, the equilibrium rationing ratio is much larger than that under two-price menu. By splitting off the time-preference effect and risk-averse effect
and assuming the random rationing rule, we obtain the similar necessary and sufficient conditions on the seller’s capacity constraint and on the shape of the demand curve.

Rationing-based price discrimination can be and is often combined with classical price discrimination strategies. The second extension in this paper illustrates an example where a rationing strategy further increases the seller’s profit when integrating third degree price discrimination in the model. We consider markets where a monopoly seller charges different prices to different distinguishable consumer groups. For example, in many markets students and seniors often enjoy discount prices by showing their identification. Rationing at the discount price can further increase sellers’ profits in those markets if a one-way resale market is allowed, wherein high-priced products are resalable but not those purchased at discount prices. Although some high-valuation consumers are eligible to buy at discount prices, they are crowded out by the rationing and buy at higher prices. As a result, sales of the high-priced products increase, as do the sellers’ profits.

This paper illustrates a new tack in second degree price discrimination. Classical second degree price discrimination models rely on the correlation between consumers’ valuations and their other preferences. For example, Deneckere and McAfee (1996) and Anderson and Dana (2005) assume consumers’ valuations depend on their preferences for quality; Dana (1998) assumes consumers’ valuations depend on how risk averse they are. Although sellers cannot tell consumers apart directly, they can differentiate products along those preferences and consequently consumers will self-select based on their valuations. In our paper, consumers differ only in their private valuations. Thus, it is ineffective for the seller to differentiate products along classical dimensions. Instead, the
availability of the products becomes a new dimension, as consumers with different valuations have different attitudes about the products with different availabilities.

This paper differs from most current literature on rationing. Some researchers, such as Degraba (1995) and Gilbert and Klemperer (2000), explain rationing through demand uncertainty, i.e., that ex ante either consumers do not know their own valuations, or the seller does not know the aggregate demand. Some, such as Allen and Faulhaber (1991), Slade (1991), Haddock and McChesney (1994), and Becker (1991) explain rationing by introducing externality effects of individual purchase behaviors. So far as we know, two papers on rationing, DeGraba and Mohammed (1999) and Cayseele (1991), are related to price discrimination. Both of these papers are based on the assumption of discrete consumer valuation, while our paper uses a general demand function and provides a thorough analysis of how the shape of the demand curve affects the profitability of price discrimination. Specifically, DeGraba and Mohammed (1999) show that a seller can increase his profit by initially selling goods only in bundles and subsequently selling remaining units separately. Rationing happens in separate sales and increases the seller’s profit in bundling sale. In their model, the effect of rationing on the seller’s profit is mixed with that of bundling. Our paper emphasizes the profitability of a rationing strategy. Cayseele (1991) studies rationing-based inter-temporal price discrimination using a model where consumers’ valuations are either high or low. His model is a good example of the first extension in our paper.

The rest of the paper is organized as follows. In Section 2, we study the base model where a monopoly seller sells through a two-price menu instead of selling at a uniform price. Rationing happens at the low-price. We also present the implications of
the seller’s capacity constraint and the shape of the demand curve. In Section 3, we study a two-period model where a monopoly seller sells at a high price in the advance period, and sells the remaining products at a low price in the spot period. Rationing happens in the spot period. In Section 4, we consider a model where a monopoly seller sells at different prices to two distinguishable consumer groups. Rationing happens among the consumer groups at the lower price. Section 5 concludes this paper.

2. Rationing based Two-Price Menu

In this section, we compare a monopoly seller’s two potential selling strategies. A stylized selling strategy is to sell at a uniform price, the *monopoly price*. An alternative is to sell through a two-price menu, i.e., a *high price* and a *low price*, with quantity limits at each price. We assume the seller cannot tell consumers apart. All consumers with valuations greater than or equal to the low-price have the same right and chance to buy the product at the low-price. Consumers with valuations greater than or equal to the high-price would like to buy at the high-price if and only if they failed to buy the products at the low-price.

When the supply at the low-price is less than the demand, † products are rationed among all potential consumers. The rationing rule determines the order in which consumers’ demands are met. In this paper, we assume that there are no resale activities between consumers and adopt the random rationing rule. This assumption is consistent

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† If there is sufficient supply at the lowest price level, no one will buy at a higher price. It’s essentially uniform pricing.
with markets such as airlines and hotels, where heterogeneous consumers with unit
demands queue in random order and resale is prohibited.

Consider a monopoly seller selling a homogeneous product to consumers with a
mass normalized as one. Each consumer has a unit demand and has a privately known
valuation, \( v \), of the product. The distribution of \( v \) in the market follows \( F(v) \) between
\([v, \bar{v}]\), which is common knowledge to both the seller and all consumers. For simplicity,
we assume that the seller’s marginal production cost of the product is zero, for sale below
a capacity limit \( T \). It is not possible to sell more than \( T \) units.

Set \( p^T \) as the price to sell out all the seller’s capacity, \( T = 1 - F(p^T) \). Define the
unconstrained monopoly price \( p^{um} \) as the seller’s optimal uniform price without any
capacity constraint, \( p^{um} = \arg \max_p p(1 - F(p)) \). Define the constrained monopoly
price \( p^{cm} \) as the seller’s optimal uniform price under the capacity constraint \( T \),
\[ p^{cm} = \arg \max_p p(1 - F(p)) \quad \text{s.t.} \quad 1 - F(p) \leq T. \]

Then, according to the seller’s capacity and the demand curve, under uniform pricing,
there are three possible cases, as shown in Figure 1:
Figure 1

1) \( p_{cm}^* = p^T > p_{um}^* \) or, equivalently, \( 1 - F(p_{um}^*) > T = 1 - F(p_{cm}^*) \). The capacity constraint does not support the seller to sell at the unconstrained monopoly price. \( p_{cm}^* \) is the highest price to sell out all his capacity;

2) \( p_{cm}^* > p^T > p_{um}^* \) or, equivalently, \( 1 - F(p_{um}^*) > T > 1 - F(p_{cm}^*) \). The capacity constraint does not support the seller to sell at the unconstrained monopoly price. \( p_{cm}^* \) is a local optimal price level but not all capacity is sold out at \( p_{cm}^* \);

3) \( p^T < p_{cm}^* = p_{um}^* \) or, equivalently, \( T \geq 1 - F(p_{um}^*) \). The capacity constraint is more than the demand at the unconstrained monopoly price.

We first study the case (1) where the seller faces a binding capacity and then discuss the cases (2) and (3) where the seller has excess capacity.

Analysis on Binding Capacity
Under uniform pricing, the seller’s profit is \( \pi^0 = p^{cm}T = p^{cm}\left(1 - F\left(p^{cm}\right)\right) \).

The seller may consider a two-price menu \((p_1, t_1; p_2, t_2)\), where \(p_1\) and \(p_2\) are the two price levels, \(p_1 < p_2\), and \(t_1\) and \(t_2\) are the capacities allocated to \(p_1\) and \(p_2\), respectively, \(t_1 < 1 - F\left(p_1\right)\) and \(t_1 + t_2 \leq T\). (1)

According to the random rationing rule, the possibility of a consumer with \(v \geq p_1\) to buy a product at \(p_1\) is \(\frac{t_1}{1 - F\left(p_1\right)}\). Therefore, the demand at \(p_2\) is

\[
\left(1 - \frac{t_1}{1 - F\left(p_1\right)}\right)\left(1 - F\left(p_2\right)\right),
\]

which is equal to the amount of consumers who have valuations \(v \geq p_2\) but fail to buy the products at the low-price.

In order to maximize his profit, the seller never provides an insufficient supply at the high price \(p_2\). Otherwise, he can increase his profit by raising \(p_2\) and keeping the same supply quantity. Thus, we assume the supply equals\(^*\) to the demand at \(p_2\).

\[
t_2 = \left(1 - \frac{t_1}{1 - F\left(p_1\right)}\right)\left(1 - F\left(p_2\right)\right)
\]  (2)

Hence, the seller’s optimization problem is defined as the following:

\[
\max_{p_1, t_1; p_2, t_2} p_1 \cdot t_1 + p_2 \cdot t_2 \quad \text{s.t. (1) and (2)}
\]

Set \((p'^1_1, t^1_1; p'^1_2, t^1_2)\) as the solution of the above optimization problem and \(\pi^1 = p'^1_1 \cdot t^1_1 + p'^1_2 \cdot t^1_2\) is the corresponding profit.

\(^*\) If the seller provide excess supply at \(p_2\), it is identical to the case \(t_1 + t_2 < T\).
**Proposition 1**: When the seller faces a binding capacity under uniform pricing, he can earn more profit using a rationing-based two-price menu, if and only if there are \( p_1 < p^{cm} \) and \( p_2 > p^{cm} \), satisfying

\[
\frac{F(p^{cm}) - F(p_1)}{(p^{cm} - p_1)(1 - F(p^{cm}))} > \frac{F(p_2) - F(p_1)}{(p_2 - p_1)(1 - F(p_2))}.
\]

Through the two-price menu, the seller gains less profit from the capacity sold at the low-price \( p_1 \), compared with uniform pricing, since \( p_1 \) is less than the constrained monopoly price \( p^{cm} \). However, the lower price attracts an abundance of low-valuation consumers. High-valuation consumers are crowded out in the rationing and have to buy at the high-price \( p_2 \), which is higher than \( p^{cm} \). Consequently, the seller can gain more profit from the remaining capacity. The profitability of a rationing-based two-price menu depends on the comparison of profit loss at the low-price and profit gain at the high-price. The practical importance of Proposition 1 lies in that, if the seller knows the realization of value distribution \( F() \), Proposition 1 offers an effective method for the seller to determine whether to adopt monopoly pricing or two-price menu pricing.

It is clear from Proposition 1 that the shape of the demand curve, i.e. \( F() \), determines the profit loss at the low price, as well as the residual demand curve after rationing — consequently, the additional profit gain at the high price. To better understand the impact of \( F() \) on the effectiveness of rationing-based price discrimination, we further derive the following three corollaries.

**Corollary 1** (Necessary condition): If the distribution function has a monotonic increasing hazard rate (IHR), the rationing-based two-price menu is not profitable.
Many papers on price discrimination assume that demand distribution is either normal, or uniform. Both distributions satisfy the IHR property. This corollary shows those assumptions will not be innocuous.

**Corollary 2** (Sufficient condition 1): The seller can increase his profit through a rationing-based two-price menu if there is \( p_2 > p^{cm} \), satisfying

\[
F(p_2) - F(p^{cm}) < \frac{1}{\left( p_2 - p^{cm} \right) f(p^{cm}) + \frac{1}{1 - F(p^{cm})}}.
\]

The left side of the inequality is the demand between \( p^{cm} \) and \( p^2 \). If the demand between the constrained monopoly price and a higher price (the high-price) is sufficiently small, then the rationing-based two-price menu will be more profitable. Intuitively, Corollary 2 focuses on when the seller can obtain a large supply gain from the high-price sale. The seller can cut the constrained monopoly price a little bit and introduce rationing. Among those consumers remaining after rationing, the fewer consumers there are with valuations between the constrained monopoly price and the high price, the more consumers there are with valuations higher than the high price. As a result, the seller can gain additional profit at \( p_2 \).

**Corollary 3** (Sufficient condition 2): The seller can increase his profit by initiating a rationing-based two-price menu if there is a \( p_1 < p^{cm} \), satisfying

\[
\frac{1}{p^{cm} - p_1} > \frac{f(p^{cm})}{1 - F(p^{cm})} \quad \text{and} \quad F(p^{cm}) - F(p_1) > \frac{1}{\left( p^{cm} - p_1 \right) f(p^{cm}) + \frac{1}{1 - F(p^{cm})}}.
\]
If the demand between the constrained monopoly price and a lower price (the low-price) is larger than an upper limit, the rationing-based two-price menu will be more profitable. Intuitively, Corollary 3 focuses on the cost of rationing. By cutting the price from the constrained monopoly price to the low price, the seller faces a profit loss. Only when the seller attracts a sufficiently large number of additional consumers into the market can the rationing force most high-valuation consumers to the high price and provide sufficient profit surplus to cover the former loss.

The sufficient conditions in Corollaries 2 and 3 often hold in existing models (Shugan & Xie 2001, Geng, Wu & Whinston 2007) with the assumption of a discrete distribution function representing consumer valuation. Rationing appears as expected in those models. In practice, markets are often composed of multiple consumer groups. While the demand of each consumer group may follow a distribution function with a monotonic increasing hazard rate, the aggregate demand of the whole market may satisfy the sufficient conditions specified in Corollaries 2 and 3. If the sellers are not able to distinguish those consumer groups, they often use a rationing-based price discrimination scheme to maximize their profits. For instance, high-income and low-income consumers have different demand curves for air travel. However, airline companies cannot discriminate against high-income consumers directly, as consumers’ income information is unobservable. Rationing-based multiple-price menus are thus prevailing solutions in the airline industry. The following numerical example illustrates the extra profit from a

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§ Current literature (Dana 1998) explains this phenomenon by assuming the heterogeneity among the early consumers and later consumers. The results of this paper are not in conflict with the current literature. Our work focuses on the price discrimination ability based on the shape of the demand curve. Rationing-based price discrimination strategies can and are often combined with other price discrimination strategies. In section 4, we show an example.
rationing-based price discrimination strategy when a market is composed of two consumer groups.

**Numerical Example:**

Assume a cruise dealer has 50 seats to sell to 100 potential customers. He knows that 70% of his potential customers are from middle class families, while the remaining 30% are from low-income families. The low-income customers’ willingness to pay follows a normal distribution $N(500,100)$, while the middle-class customers’ willingness to pay follows a normal distribution $N(1000,100)$. If the seller sells at a uniform price, his optimal price is $560, and his expected profit is $28,000. If he uses a two-price menu, he sells 38 seats at $490 and 12 seats at $860. His expected profit, consequently, increases to $28,940. The following figure shows the difference between the two strategies.

![Figure 2](image-url)
Analysis of Excess Capacity

The seller may have excess capacity under uniform pricing, as described in cases (2) and (3) above.

Proposition 2: If \( p^{cm} > p^T > p^{um} \) or, equivalently \( 1 - F(p^{um}) > T > 1 - F(p^{cm}) \), then the seller can increase his profit by adopting a rationing-based two-price menu.

In case (2), the seller would rather waive some of his capacity under uniform pricing, as at the price to sell out all his capacity, he gains even less profit than that at \( p^{cm} \). However, noticing \( p^{cm} > p^{um} \) and \( \pi(p^{cm}) < \pi(p^{um}) \), there must be a very large demand between \( p^{cm} \) and \( p^{um} \). Thus, by setting the low-price at \( p^{um} \), the seller can attract sufficient additional consumers without losing too much marginal profit. Meanwhile, he can set a high-price above \( p^{cm} \) to skim off a larger surplus from the high-valuation consumers crowded out from the rationing. A rationing-based two-price menu helps sellers take advantage of the shape of the demand curve and to increase profit.

Proposition 3: If \( p^T > p^{cm} = p^{um} \) and \( T \geq 1 - F(p^{um}) \), the seller cannot earn more profit from a rationing-based two-price menu than from uniform pricing.

In case (3), the seller has a sufficiently large capacity to support the unconstrained monopoly price. Under this situation, uniform pricing at the unconstrained monopoly price is more profitable than any rationing-based two-price menu. In other words, \( \pi(p^{cm}) = \pi(p^{um}) = p^{um} \left( 1 - F(p^{um}) \right) \) is the maximum monopoly profit.
Analysis of Social Welfare

**Proposition 4**: Under a rationing-based two-price menu selling strategy, the total social welfare is less than that under a uniform pricing strategy.

Intuitively, under the rationing-based two price menu, some consumers with valuations less than the constrained monopoly price buy the product, while the same amount of consumers with valuations greater than or equal to the constrained monopoly price fail to buy the product. When the products are distributed to consumers with lower valuations \( v < p^{\text{cmvp}} \), the allocation becomes less inefficient; thus, the total social welfare decreases. Therefore, rationing may be the optimal strategy for the seller, but is not a social optimal solution.

3. Rationing based Premium Advance Selling

Rationing-based price discrimination has multiple formats in practice. Under the two-price menu described in the above section, the seller offers both the low price and the high price at the beginning. Consumers first try to buy at the low price; if unsuccessful, then they buy at the high price. Besides the two-price menu, the seller can sell part of his capacity in advance at a price and then sell the rest of the capacity at another price at the spot time. This is an advance selling strategy, a common intertemporal price discrimination strategy.

There are two types of advance selling strategies: *discount advance selling*, where the advance price is lower than the spot price, and *premium advance selling*, where the
advance price is higher than the spot price. Discount advance selling is essentially the same as the two-price menu strategy. In this section, we study premium advance selling, where rationing also frequently occurs. For example, a box office will sell concert tickets several months before the concert at the face price plus a significant convenience fee, while holding back a certain number of tickets to sell at the face value right before the concert to the long queue in front of the theater.

To explain rationing-based premium advance selling, we use a setup similar to that described in Section 2, except with the following differences. First, there are two periods, the advance period and the spot period. A monopoly seller can sell his products in either or both periods. Second, we assume that the seller in the advance period can commit to the price and the quantity supplied for the spot sale. This assumption is consistent with many markets, such as event ticket markets. Third, all consumers enter the market in the advance period and completely know the seller’s selling strategy. They can decide to either buy in the advance period or strategically wait for the spot period. Fourth, all consumers are risk neutral.**

Similar to the structure of the scenario in Section 2, we use $p^T$ to represent the price to sell out all the seller’s capacity, $T = 1 - F\left(p^T\right)$; $p^{um}$ to represent the seller’s optimal uniform price without any capacity constraint, $p^{um} = \arg \max_p p\left(1 - F\left(p\right)\right)$; and $p^{cm}$ to represent the seller’s optimal uniform price under the capacity constraint $T$,

$$p^{cm} = \arg \max_p p\left(1 - F\left(p\right)\right) \text{ s.t. } 1 - F\left(p\right) \leq T.$$  

** If consumers are risk averse, the rationing based premium advance selling will be more profitable. In this paper we adopt the risk neutral assumption because we would like to separate profitability of rationing based price discrimination to that from risk aversion.
Then, according to the seller’s capacity and the demand curve, under uniform pricing, there are three possible cases:

1) \( p^{cm} = p^T > p^{um} \) or, equivalently, \( 1 - F(p^{um}) > T = 1 - F(p^{cm}) \). The capacity constraint does not allow the seller to sell at the unconstrained monopoly price. \( p^{cm} \) is the highest price to sell out all his capacity;

2) \( p^{cm} > p^T > p^{um} \) or, equivalently, \( 1 - F(p^{um}) > T > 1 - F(p^{cm}) \). The capacity constraint does not allow the seller to sell at the unconstrained monopoly price. \( p^{cm} \) is a local optimal price level, but not all the capacity is sold out at \( p^{cm} \);

3) \( p^T < p^{cm} = p^{um} \) or, equivalently, \( T \geq 1 - F(p^{um}) \). The capacity constraint is more than the demand at the unconstrained monopoly price.

We first study case (1) where the seller faces a binding capacity and then discuss cases (2) and (3) where the seller has excess capacity.

In case (1), as \( T = 1 - F(p^{cm}) \), the seller’s profit under uniform pricing is

\[
\pi^0 = p^{cm} \left(1 - F(p^{cm})\right).
\]

Analysis of Binding Capacity

We consider a premium advance selling strategy \((p_1, t_1; p_2, t_2)\), where the seller sells \( t_1 \) products at \( p_1 \) in the advance period and \( t_2 \) products at \( p_2 \) in the spot period, \( p_1 > p_2 \) and \( t_1 + t_2 \leq T \).  (1)
Given \( p_1 > p_2 \), if a consumer with a valuation \( v \) waits to buy at the spot period, her surplus is \( r(v - p_2) \), where \( r \) denotes the probability that she buys the product in the spot period through rationing. If there is no rationing in the second period, \( r = 1 \); otherwise, \( r \in (0,1) \). If the consumer buys a product in the advance period, her surplus is \( v - p_1 \). Thus the consumer would like to buy in advance if and only if
\[
v - p_1 \geq r(v - p_2). \tag{3}
\]

If \( r = 1 \), all consumers would strategically wait to buy at the spot time, since \( v - p_1 < v - p_2 \). It is then essentially identical to the uniform pricing strategy. We study the situation where \( r < 1 \).

**Lemma 1:** If there is rationing in the spot period, then there exists a threshold level \( v^* \) such that all consumers with \( v \in [v^*, \tilde{v}] \) would like to buy in the advance period, and consumers with \( v \in [p_2, v^*) \) would like to buy through rationing in the spot period.

Lemma 1 shows that consumers self-select because of the rationing in the spot period. Although rationing in the spot period does not affect the product quality, it does reduce the expected surplus consumers gain. The availability becomes another factor in addition to the price affecting consumers’ decisions. Consumers with valuations higher than \( v^* \) would rather buy in the advance period, as expected rationing in the spot period reduces their expected surplus dramatically.

Since consumers with \( v \in [p_2, v^*) \) would like to buy in the spot period, according to the random rationing rule, the possibility of buying a unit of the product then is
Then, for the marginal consumers between those buying in the advance period and those buying in the spot period,

\[ v^* - p_1 = \frac{t_2}{F(v^*) - F(p_2)} (v^* - p_2) \quad (4) \]

In order to maximize his profit, the seller never provides an insufficient supply at the high-price \( p_1 \). Otherwise, he can increase his profit by raising \( p_1 \) and keeping the same supply quantity. Thus, we assume the supply equals\(^\dagger\) the demand at \( p_1 \).

\[ t_1 = 1 - F(v^*) \quad (5) \]

The seller’s optimal pricing strategy is defined as the following

\[
\max_{p_1, t_1; p_2, t_2} \quad p_1 \cdot t_1 + p_2 \cdot t_2 \quad \text{s.t.} \ (1), \ (4) \text{ and } (5) \text{ hold.}
\]

Set \( (p_1^2, t_1^2; p_2^2, t_2^2) \) as the solution of the above optimization problem, and \( \pi^2 \equiv p_1^2 \cdot t_1^2 + p_2^2 \cdot t_2^2 \) is the corresponding profit.

**Proposition 5:** When the seller faces a binding capacity under uniform pricing, he can increase his profit by adopting a rationing-based premium advance selling strategy if and only if there exists \( p_2 < p^{cm} < v^* \), s.t.

\[
\frac{F(p^{cm}) - F(p_2)}{(p^{cm} - p_2)(1 - F(p^{cm}))} > \frac{F(v^*) - F(p_2)}{(v^* - p_2)(1 - F(v^*))}.
\]

\(^\dagger\) If the seller provide excess supply at \( p_1 \), it is identical to the case \( t_1 + t_2 < T \)
Under the premium advance selling strategy, the seller cuts the spot period price lower than the constrained monopoly price and artificially introduces rationing in that period. His profit from the spot period is reduced by the rationing strategy; however, rationing forces high-valuation consumers to buy in the advance period at a much higher price and consequently increases the seller’s profit in the advance period. The seller would like to introduce rationing in the spot period if the profit gain in the advance period outweighs the profit loss in the spot period. The differences in the profits in the two periods are determined by the shape of the demand curve.

To conduct the analysis in an easier way, we first show an interesting result in a comparison of the rationing-based premium advance and two-price menu strategies.

**Lemma 2:** For a given demand curve, the seller gains the same profit under the optimal rationing-based premium advance selling strategy and the rationing-based two-price menu selling strategy.

Under the two-price menu selling strategy, the seller is not able to discriminate all high-valuation \( v \geq p^1_2 \) consumers, as some of them get the low-price products during the rationing. Under the premium advance selling strategy, the seller can discriminate all high-valuation \( v \geq v^* = p^1_2 \) consumers. But he can only charge them at \( p^2_2 < v^* \). By assuming random rationing rule and that consumers are risk neutral and, the failures to further skim high-valuation consumers’ surplus are the same under the two rationing strategies.

Based on Lemma 2, we have Corollaries 4, 5, 6.
Corollary 4 (Necessary condition): If the distribution function has a monotonic increasing hazard rate, rationing-based premium advance selling is not profitable.

Corollary 5 (Sufficient condition 1): The seller can increase his profit through rationing-based premium advance selling if there is \( v^* > p^{cm} \), satisfying

\[
F(v^*) - F(p^{cm}) < \frac{1}{\frac{1}{v^* - p^{cm}} f(p^{cm}) + \frac{1}{1 - F(p^{cm})}}.
\]

Corollary 6 (Sufficient condition 2): The seller can increase his profit through rationing-based premium advance selling if there is a \( p_2 < p^{cm} \), satisfying

\[
\frac{1}{p^{cm} - p_2} > \frac{f(p^{cm})}{1 - F(p^{cm})} \quad \text{and} \quad F(p^{cm}) - F(p_2) > \frac{1}{\frac{1}{p^{cm} - p_2} f(p^{cm})} - \frac{1}{1 - F(p^{cm})}.
\]

Analysis of Excess Capacity

Similarly, we have Propositions 6 and 7 addressing cases (2) and (3) above.

Proposition 6: If \( p^{cm} > p^T > p^{um} \) and \( 1 - F(p^{um}) > T > 1 - F(p^{cm}) \), then the seller can increase his profit by adopting rationing-based premium advance selling.

Proposition 7: If \( p^T > p^{cm} = p^{um} \) and \( T \geq 1 - F(p^{um}) \), the seller cannot earn more profit from a rationing-based premium advance selling strategy than from a uniform pricing strategy.
Analysis of Social Welfare

Proposition 8: Under the rationing-based premium advance selling strategy and the
rationing-based two-price menu selling strategy, the total social welfare is the same, but
less than that under optimal uniform pricing.

Intuitively, under both rationing strategies, consumers with valuation $v \in [p_1^2, v^*)$ buy the products at $p_1^2$ with the same odds during the rationings.\footnote{Remember $p_1^1 = p_2^2$ and $v^* = p_2^1$. See the proof of Proposition 3.} All consumers with high valuation $v \geq v^*$ eventually buy the products. Therefore, the social welfare is the same under the two price discrimination strategies.

4. Rationing-based Price Discrimination with Distinguishable Consumer Groups

Different from classical price discrimination strategies, rationing-based price
discrimination is based on the shape of the demand curve but not on the heterogeneity of
consumers or products. It provides sellers with a new strategy to further increase their
profits. It can be, and has often been, combined with other price discrimination strategies.
In this section, we introduce an example where rationing-based price discrimination is
combined with a typical third degree price discrimination strategy.

We consider markets where the demand comes from several distinguishable
consumer groups. The seller can distinguish between groups, yet not between members of
the same group. For instance, many NCAA football teams sell tickets to college students
at discounted prices and to non-student fans at regular prices. Besides the typical third-
degree price discrimination in those markets, sellers would also like to adopt a rationing
strategy if there is a *one-way resale market* between the market segments — from the high-priced segment to the low-priced segment, *but not vice versa*. The one-way resale market can be easily realized. In the above example, discounted student tickets are always forbidden to be resold to non-student fans, but regular tickets are allowed to be resold to students.

We assume consumers are characterized as \((x, v)\), where \(x\) is the characteristic (signal) observable to the seller, \(x = \{x_1, x_2\}\). The market share of consumers with \(x = x_1\) is \(\alpha\) and that of consumers with \(x = x_2\) is \(1 - \alpha\). \(v\) is the consumer’s valuation on the product. \(v\) follows the distribution \(F_x(v)\) between \([v_1, v]\). \(F_{x_1}(\cdot)\) and \(F_{x_2}(\cdot)\) are common knowledge to both the seller and the consumers. The seller faces a capacity constraint \(T\).

He divides the market into two segments according to consumers’ observable characteristic, \(x\). He sells \(t_1\) products at \(p_1\) to the consumers with \(x = x_1\), namely *market segment 1*; and sells \(t_2\) products at \(p_2\) to the consumers with \(x = x_2\), namely *market segment 2*, \(t_1 + t_2 \leq T\). (6)

If there is no rationing in either segment, \(t_1 = \alpha \left(1 - F_{x_1}(p_1)\right)\) and \(t_2 = (1 - \alpha) \left(1 - F_{x_2}(p_2)\right)\). The seller’s optimal selling strategy is defined as the following

\[
\max_{p_1, p_2} \alpha p_1 \left(1 - F_{x_1}(p_1)\right) + (1 - \alpha) p_2 \left(1 - F_{x_2}(p_2)\right),
\]

\[
\text{s.t. } \alpha \left(1 - F_{x_1}(p_1)\right) + (1 - \alpha) \left(1 - F_{x_2}(p_2)\right) \leq T.
\]
Set \( (p_1^3, t_1^3; p_2^3, t_2^3) \) as the optimal solution without rationing. The optimal segment prices depend on the respective demand elasticity in the segments. The more elastic the demand is, the lower the segment price is.

\[
\frac{p_2^3}{p_1^3} = \frac{1 - \frac{1}{e_1}}{1 - \frac{1}{e_2}}, \text{ where } e_1 \text{ and } e_2 \text{ are the respective price elasticities in the two market segments, } e_i = \frac{f(p_i^3)p_i^3}{1 - F_i(p_i^3)}, i = 1, 2. \text{ Without loss of generality, we assume } p_1^3 < p_2^3.
\]

Now, we consider an adjustment on the above third degree price discrimination strategy by introducing rationing. Instead of choosing \( (p_1^3, t_1^3; p_2^3, t_2^3) \), the seller may consider a rationing strategy \( (p_1^3 - \varepsilon, t_1^3 - \Delta t; p_2^3, t_2^3 + \Delta t) \), where \( t_i^3 - \Delta t < 1 - F_i(p_i^3 - \varepsilon) \).

Or, rationing happens in market segment 1, where the price is lower than it is in the other market segment.

As the supply is scarce, consumers with \( v \geq p_1^3 - \varepsilon \) in market segment 1 have a chance of

\[
\frac{\alpha[F_i(p_1^3) - F_i(p_1^3 - \varepsilon)] + \Delta t}{\alpha[1 - F_i(p_1^3 - \varepsilon)]} \text{ to not get a product at } p_1^3 - \varepsilon. \text{ Among those unlucky consumers, }
\]

\[
\frac{1 - F_i(p_r)}{1 - F_i(p_1^3 - \varepsilon)}\left[\alpha(F_i(p_1^3) - F_i(p_1^3 - \varepsilon)) + \Delta t\right] \text{ have a willingness to buy at } p_r \text{ from market segment 2 through the one-way resale market, where } p_r \text{ is the market clearing resale price.} \]
Given the segment prices $p_1^3 - \varepsilon$ and $p_2^3$, if $p_r > p_2^3$, all consumers with $x_2$ would like to buy and resell products for arbitrage profit. Competition between resellers will lower the resale market price. If $p_r < p_2^3$, no one will like to buy and resell products, and a resale market does not exist. Therefore, $p_r = p_2$.

Thus, through the one-way resale market, rationing in market segment 1 creates an extra demand
\[
\frac{1 - F_{x_1}(p_2^3)}{1 - F_{x_1}(p_1^3 - \varepsilon)} \left[ \alpha \left( F_{x_1}(p_1^3) - F_{x_1}(p_1^3 - \varepsilon) \right) + \Delta t \right]
\]
for sales in the market segment 2.

In order to maximize his profits, the seller will provide a sufficient supply at $p_2$ to meet all the potential demand, including both the demand from the second market segment and the demand from the first market segment through the one-way resale market. Thus, set $\Delta t = \alpha [1 - F_{x_1}(p_2^3)] - F_{x_1}(p_1^3 - \varepsilon) / F_{x_1}(p_2^3) - F_{x_1}(p_1^3 - \varepsilon)$, then the extra supply the seller adds to market segment 2 equals the additional demand created by rationing in market segment 1, or
\[
\Delta t = \left[1 - F_{x_1}(p_2^3)\right] \frac{\alpha \left[F_{x_1}(p_1^3) - F_{x_1}(p_1^3 - \varepsilon)\right] + \Delta t}{\left[1 - F_{x_1}(p_1^3 - \varepsilon)\right]}.
\]

Given the above changes on the prices and capacity allocation, the change in the seller’s profit is $\Delta \pi = \Delta t(p_2^3 - p_1^3 + \varepsilon) - t^3 \varepsilon$. A rationing strategy is profitable if and only if $\Delta \pi > 0$.

**Proposition 9:** The seller can increase his profit by inducing an arbitrary rationing in market segment 1, if $f_{x_1}(p_1^3)/(1 - F_{x_1}(p_1^3)) > \frac{F_{x_1}(p_2^3) - F_{x_1}(p_1^3)}{p_2^3 - p_1^3}/(1 - F_{x_1}(p_1^3))$. 

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Without rationing, the seller’s selling strategy is typical third-degree price discrimination. The seller introduces rationing in the market segment with the lower price and forces some high-valuation consumers in this segment to buy through the resale market. The price discrimination mechanism is a mix of second-degree price discrimination and third-degree price discrimination.

It is worth noting that in reality, sellers do not necessarily rely on the one-way resale market to realize the extra profit in rationing. Instead, one common variation is that the seller allows all consumers to buy at the higher price $p_2$, but only allow consumers with $x_1$ to buy at the lower price $p_1$. For instance, students, as a consumer group with an average low income, enjoy special discounts in many markets. However, they often face limited supply. If students with high valuations fail to buy with the student discounts, they can also buy at the regular prices that apply to non-student consumers.

5. Conclusion

This paper provides a theory of rationing where rationing functions as an effective mechanism for second-degree price discrimination. Rationing at the low-price sale creates availability heterogeneity among the products, by which high-valuation consumers are forced to procure them at the high price. A new vehicle is shown to distinguish consumers, the availability of a product. The seller can conduct a second-degree price discrimination strategy by artificially controlling the products’ availabilities. However, the control is under the constraint of the seller’s capacity and the shape of the
demand curve. By a more general definition of “quality,” availability may also be treated as a special dimension of quality. In this sense, our work on rationing-based price discrimination is consistent with the classical literature on second degree price discrimination.

Rationing-based price discrimination can be implemented in many different ways. Besides the base model where two prices are offered at the same time, the seller can offer prices in a sequential order, as shown in Section 3, or can combine rationing-based price discrimination with other price discrimination strategies, as shown in Section 4.

This research on rationing-based price discrimination can be extended in a number of directions. First, this paper assumes the random rationing rule in all models, in order to simplify the analysis and to solely show the intuition of the rationing. Consumer heterogeneity on rationing costs leads to various rationing rules in practice. Research based on the general rationing rule may provide a more complete view of rationing’s potential role in price discrimination. Second, the current model assumes the demand curve is well known to the seller. It would be interesting to examine how uncertainty in demand affects the rationing-based price discrimination strategy. Third, besides the example shown in Section 4, more cases are worth studying where rationing-based price discrimination is combined with other price discrimination strategies.
Appendix

Proof to Proposition 1

Consider the optimal two-price selling strategy \((p_1, t_1; p_2, t_2)\)

If \(t_1 + t_2 < T\), (1) is not binding, substitute the (2) into the seller’s optimization problem

\[
\pi' = \max_{p_1, p_2} p_1 t_1 + p_2 \left(1 - \frac{t_1}{1 - F(p_1)}\right)\left(1 - F(p_2)\right)
\]

\[
\frac{\partial \pi}{\partial t_1} = p_1 - \frac{1 - F(p_2)}{1 - F(p_1)} p_2
\]

If \(p_1 \neq p_2 \frac{1 - F(p_2)}{1 - F(p_1)}, \frac{\partial \pi}{\partial t_1} \neq 0\). Then \(t_1 = 0\) or \(\min\{1 - F(p_1), T\}\). There is no rationing. The two-price selling strategy is identical to uniform pricing. If

\[
p_1 = p_2 \frac{1 - F(p_2)}{1 - F(p_1)}, t_1\text{ can be any value between 0 and } \min\{1 - F(p_1), T\}. \text{ The value of } t_1\text{ does not affect the seller’s profit. Set } t_1 = 0, \text{ then the seller’s profit under two-price selling strategy }\(p_1, t_1; p_2, t_2)\text{ is no more than that under uniform pricing. Therefore, if (1) is not binding, the seller cannot increase his profit by rationing-based two-price selling strategy.}
\]

We then consider the case when (1) is binding. That is, \(t_1 + t_2 = T = 1 - F(p_{\text{cmp}})\).

Recall \(t_2 = \left(1 - \frac{t_1}{1 - F(p_1)}\right)\left(1 - F(p_2)\right), \text{ thus,}\)
\[ t_1 = \frac{F(p_2) - F(p_{cm})}{F(p_2) - F(p_1)} (1 - F(p_1)) \quad \text{and} \quad t_2 = \frac{F(p_{cm}) - F(p_1)}{F(p_2) - F(p_1)} (1 - F(p_2)) \]

Compare the seller’s profit under the two-price menu and that under the monopoly price \( p_{cm} \), \( \Delta \pi = t_2 (p_2 - p_{cm}) - t_1 (p_{cm} - p_1) = t_2 (p_2 - p_1) - T(p_{cm} - p_1) \).

Rationing-based two-price menu is profitable if and only if \( \Delta \pi > 0 \), or

\[ \frac{F(p_{cm}) - F(p_1)}{F(p_2) - F(p_1)} (1 - F(p_2)) (p_2 - p_1) > (1 - F(p_{cm})) (p_{cm} - p_1) \]

That is,

\[ \frac{F(p_{cm}) - F(p_1)}{(p_{cm} - p_1)(1 - F(p_{cm}))} > \frac{F(p_2) - F(p_1)}{(p_2 - p_1)(1 - F(p_2))} \]

**Proof to Corollary 1:**

In the following analysis we adopt the notations in Proposition 1. Yet all results can apply to conclusions related to Proposition 2 as well.

From Proposition 1, rationing is profitable if and only if

\[ \frac{F(p_{cm}) - F(p_1)}{(p_{cm} - p_1)(1 - F(p_{cm}))} > \frac{F(p_2) - F(p_1)}{(p_2 - p_1)(1 - F(p_2))} \]

Given any \( p_1 < p_{cm} \), set \( g(p) = \frac{F(p) - F(p_1)}{(p - p_1)(1 - F(p))} \).

If \( g'(p) \geq 0 \) for \( p \geq p_{cm} \), then for any \( p_2 > p_{cm} \), \( g(p_{cm}) \leq g(p_2) \), rationing-based price discrimination strategy is not profitable.
\[
g'(p) = \frac{f(p)(p-p_1)(1-F(p)) - (F(p) - F(p_1))(1-F(p) - f(p)(p-p_1))}{(p-p_1)(1-F(p))^2}
\]

\[
= \frac{f(p)(p-p_1)(1-F(p_1)) - (F(p) - F(p_1))(1-F(p))}{(p-p_1)(1-F(p))^2}
\]

Thus, \(g'(p) \geq 0\) if and only if \(f(p)(p-p_1)(1-F(p_1)) \geq (F(p) - F(p_1))(1-F(p))\)

Or \(\frac{f(p)}{1-F(p)} \geq \frac{p-p_1}{1-F(p_1)}\)

Note that there exists a \(p' \in [p_1,p]\), s.t. \(\frac{p-p_1}{1-F(p_1)} = \frac{f(p')}{1-F(p_1)} \leq \frac{f(p')}{1-F(p')}\).

If \(\frac{f(v)}{1-F(v)}\) monotonic increasing, between \(p'\) and \(p\), then \(\frac{f(p')}{1-F(p')} \leq \frac{f(p)}{1-F(p)}\).

That means \(g'(p) \geq 0\), or rationing-based price discrimination strategy is not profitable.

**Proof to Corollary 2**

Given Proposition 1, set \(p_i = p^{cm} - \varepsilon\). Then take \(\varepsilon \to 0\), we can get

\[
\frac{f(p^{cm})}{1-F(p^{cm})} \geq \frac{F(p_2) - F(p^{cm})}{(p_2-p^{cm})(1-F(p_2))}.
\]

Then, \(F(p_2) - F(p^{cm}) < \frac{1}{1} \frac{1}{(p_2-p^{cm})f(p^{cm})} + \frac{1}{1-F(p^{cm})} \)

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Proof to Corollary 3

Follow the analysis in the proof to Corollary 1, if \( g'(p)|_{p=p^{cm}} < 0 \), then set \( p_2 = p^{cm} + \epsilon \). Rationing-based price discrimination strategy is profitable. Thus, the sufficient condition is

\[
\exists p_1, \text{s.t.} \quad \frac{f(p^{cm})}{1 - F(p^{cm})} < \frac{F(p^{cm}) - F(p_1)}{p^{cm} - p_1} \frac{1}{1 - F(p_1)}
\]

That means, \( \frac{1}{p^{cm} - p_1} > \frac{f(p^{cm})}{1 - F(p^{cm})} \) and

\[
F(p^{cm}) - F(p_1) > \frac{1}{1 - F(p^{cm})} \frac{1}{1 - F(p^{cm})} \frac{1}{1 - F(p^{cm})} - \frac{1}{1 - F(p^{cm})}
\]

Proof to Proposition 2

Under the single-price selling, the seller only sells out \( 1 - F(p^{cm}) < T \) units of products. We consider a two-price menu in which \( q_1 + q_2 = 1 - F(p^{cm}) \). Then all the results and proves of the proposition 1 and the corollaries 1,2,3 still hold when comparing the profit of the above two-price menu and that of the constrained monopoly price.

Recall \( p^{um} < p^{cm} \), and \( p^{um}(1 - F(p^{um})) > p^{cm}(1 - F(p^{cm})) \)
Thus, \[
\frac{1}{\frac{p^m}{p^m}} = \frac{F\left(p^m\right) - F\left(p_1\right)}{(p^0 - p_1)(1 - F\left(p_1\right))}.
\]

As \( p^0 \) is the local optimal price, \[
\frac{f\left(p^m\right)}{1 - F\left(p^m\right)} = \frac{1}{p^m}.
\]

Therefore, \[
\frac{f\left(p^m\right)}{1 - F\left(p^m\right)} < \frac{F\left(p^m\right) - F\left(p_1\right)}{(p^m - p_1)(1 - F\left(p_1\right))}.
\]

According to analysis in the proof to the corollary 1, \( g'(p) < 0 \), then set \( p_2 = p^m + \varepsilon \), \( g\left(p_2\right) > g\left(p^m\right) \). Or, A two-price menu at \( p^m \) and \( p^m + \varepsilon \) is more profit than uniform pricing at \( p^m \), without increase any capacity. Therefore, the seller can increase his profit by adopting a rationing-based two-price menu.

**Proof to Proposition 3**

We consider the optimal two-price menu without capacity constraint. We show the profit under this selling strategy cannot be greater than \( \pi^0 \).

Without the capacity constraint, the seller’s optimization problem is rewritten as following

\[
\max_{p_1, t_1; p_2, t_2} p_1 \cdot t_1 + p_2 \cdot t_2 \quad \text{s.t.} \quad t_2 = \left(1 - \frac{t_1}{1 - F\left(p_1\right)}\right)(1 - F\left(p_2\right))
\]

or

\[
\max_{p_1, t_1; p_2} p_1 \cdot t_1 + p_2 \left(1 - \frac{t_1}{1 - F\left(p_1\right)}\right)(1 - F\left(p_2\right))
\]
F.O.C. w.r.t. \( t_1 \) is
\[
\frac{\partial \pi}{\partial t_1} = p_1 - p_2 \frac{1 - F(p_2)}{1 - F(p_1)}
\]

If \( p_1 \neq p_2 \frac{1 - F(p_2)}{1 - F(p_1)} \), \( t_1 = 0 \) or \( 1 - F(p_1) \), there is no rationing;

If \( p_1 = p_2 \frac{1 - F(p_2)}{1 - F(p_1)} \), \( t_1 \) can be any value between 0 and \( 1 - F(p_1) \), and will not change the profit. It means the profit without rationing is the maximum profit when capacity is unlimited. As \( p^{um} \) is the global optimal price, a rationing-base two-price menu is not more profitable than the optimal uniform pricing strategy.

**Proof to Proposition 4**

Under uniform pricing, all and only consumers with valuation greater than or equal to the market clearing price buy the product. The total social welfare is
\[
W^0 = \int_{p^{cm}}^{v} v dF(v).
\]

Under the rationing-based two-price menu, some consumers with valuation less than the constrained monopoly price buy the product, while a same amount of consumers with valuation greater than or equal to the constrained monopoly price fail to buy the product.

According to the proof to the proposition 1, under the optimal two-price menu,
\[
t_1 = \frac{F(p_1) - F(p^{cm})}{F(p_2) - F(p_1) (1 - F(p_1))}.
\]

Thus, the chance of a high-valuation consumer to get product in the rationing is
\[
\frac{t_1}{1 - F(p_1)} = \frac{F(p_1) - F(p^{cm})}{F(p_2) - F(p_1)}
\]
Therefore, the total social welfare is

\[ W^1 = W^0 - \int_{p_1}^{p_2} \left(1 - \frac{F(p_2^m) - F(p_1^m)}{F(p_2) - F(p_1)}\right) v dF(v) + \int_{p_1}^{p_2} \frac{F(p_2^m) - F(p_1^m)}{F(p_2) - F(p_1)} v dF(v) \]

\[ = W^0 - \frac{1}{F(p_2) - F(p_1)} \left( \int_{p_1}^{p_2} (F(p_2^m) - F(p_1^m)) v dF(v) - \int_{p_1}^{p_2} (F(p_2^m) - F(p_1^m)) v dF(v) \right) \]

\[ < W^0 - \frac{1}{F(p_2) - F(p_1)} \left( \int_{p_1}^{p_2} (F(p_2^m) - F(p_1^m)) p^m dF(v) - \int_{p_1}^{p_2} (F(p_2^m) - F(p_1^m)) p^m dF(v) \right) \]

\[ = W^0 \]

**Proof to Lemma 1:**

From (4), a consumer would like to buy in the advance period if \((1 - r)v \geq p_1 - r p_2\). The marginal consumers has the valuation \(v^* = \frac{p_1 - r p_2}{1 - r} = p_1 + \frac{r(p_1 - p_2)}{1 - r}\).

**Proof to Proposition 5:**

Uniform pricing is identical to the two-price selling strategy \((0, 0; p^m, T)\), the seller’s profit is \(\pi^0\). \(T = 1 - F(p^m)\)

Now considering a premium advance selling \((p_1, t_1; p_2, t_2)\). Similar to argument in the proof to Proposition 1, if (1) is not binding, the seller cannot earn more profit by introducing rationing. We then consider the case where (1) is binding. That is

\[ t_1 + t_2 = T = 1 - F(p^m) \]

Given the above change on the selling strategy, the change of the seller’s profit is
\[ \Delta \pi = t_1 \left( p_1 - p^{cm} \right) - t_2 \left( p^{cm} - p_2 \right) \]

\[ \Delta \pi > 0 \quad \text{iff} \quad t_1 \left( p_1 - p_2 \right) > T \left( p^{cm} - p_2 \right) \quad (7) \]

Recall \( t_1 = 1 - F(v^*) \). Thus, \( t_2 = T = t_1 = F(v^*) - F(p^{cm}) \)

From (4), \( p_1 = v^* - \frac{t_2}{F(v^*) - F(p_2)} (v^* - p_2) = v^* - \frac{F(v^*) - F(p^{cm})}{F(v^*) - F(p_2)} (v^* - p_2) \)

Therefore, (7) can be rewritten as

\[ \left( 1 - F(v^*) \right) \left( v^* - \frac{F(v^*) - F(p^{cm})}{F(v^*) - F(p_2)} (v^* - p_2) - p_2 \right) > \left( 1 - F(p^{cm}) \right) (p^{cm} - p_2) \]

or,

\[ \frac{F(p^{cm}) - F(p_2)}{(p^{cm} - p_2)(1 - F(p^{cm}))} > \frac{F(v^*) - F(p_2)}{(v^* - p_2)(1 - F(v^*))} \]

**Proof to Lemma 2:**

The seller’s maximum profit from multiple-price menu selling strategy is defined by

\[ \pi_1 = \max_{p_1, t_1; p_2, t_2} p_1 \cdot t_1 + p_2 \cdot t_2 \quad \text{s.t.} \ (1) \text{ and } (2) \text{ hold} \]

His maximum profit from the premium advance selling strategy is defined by

\[ \pi_2 = \max_{p_1, t_1; p_2, t_2} p_1 \cdot t_1 + p_2 \cdot t_2 \quad \text{s.t.} \ (1), \ (4) \text{ and } (5) \text{ hold} \]
From (1) and (2), we get \( t_1 = \frac{(1 - F(p))((T + F(p_2)) - 1)}{F(p_2) - F(p_1)} \), substitute it and (1) into the maximization problem under two-price selling strategy, we get

\[
\pi_2 = \max_{p_1, p_2} p_2 \left( T - \frac{(1 - F(p))(T + F(p_2) - 1)}{F(p_2) - F(p_1)} \right) + p_1 \frac{(1 - F(p))(T + F(p_2) - 1)}{F(p_2) - F(p_1)}
\]

\( \tag{8} \)

Given the above lemma, substitute (1), (4) and (5) into the maximization problem under advance selling strategy, we can get,

\[
\pi_1 = \max_{v^*, p_2} (1 - F(v^*)) \left[ v - \frac{1 - F(v^*)}{F(v^*) - F(p_2)} (v^* - p_2) \right] + \left( T - (1 - F(v^*)) \right) p_2
\]

\( \tag{9} \)

(8) and (9) are essentially identical, where \( v^* \) in (9) corresponds \( p_2 \) in (8), and \( p_2 \) in (9) corresponds \( p_1 \) in (8). Thus, \( \pi_1 = \pi_2 \).

**Proof to Proposition 8**

As we shown in the proof to the lemma 2, the seller’s optimization problems are identical under premium advance selling and under two-price selling. Thus, the solution of \( v^* \) in (9) equals to that of \( p_2 \) in (8), and the solution of \( p_2 \) in (9) equals to that of \( p_1 \) in (8).

Therefore, under both selling strategies, consumers with valuation greater than or equal to \( v^* \) in (9), or \( p_2 \) in (8) eventually buy the product. Consumers with valuation between \( v^* \)
and \( p_2 \) in (9), or \( p_2 \) and \( p_1 \) in (8) buy the products through rationing. As the capacity constraint keeps constant, the possibilities of later consumers buy the product are same under both advance selling strategies. Thus, the total social welfares under both advance selling strategies are the same.

**Proof to Proposition 9**

Substitute the \( \Delta t = \alpha [1 - F_{x_1}(p_2^3)] \frac{F_{x_1}(p_1^3)}{F_{x_1}(p_2^3)} - F_{x_1}(p_1^3 - \varepsilon) \) into the profit change equation

\[
\Delta \pi = \Delta t (p_2^3 - p_1^3 + \varepsilon) - t^3 \varepsilon .
\]

\( \Delta \pi > 0 \) iff \( \frac{1 - F_{x_1}(p_2^3)}{1 - F_{x_1}(p_1^3)} \cdot \frac{F_{x_1}(p_1^3) - F_{x_1}(p_1^3 - \varepsilon)}{F_{x_1}(p_2^3) - F_{x_1}(p_1^3 - \varepsilon)} (p_2^3 - p_1^3 + \varepsilon) > \varepsilon \)

Or \( \frac{F_{x_1}(p_1^3) - F_{x_1}(p_1^3 - \varepsilon)}{\varepsilon} \left/ \left(1 - F_{x_1}(p_1^3)\right)\right. > \frac{F_{x_1}(p_2^3) - F_{x_1}(p_1^3 - \varepsilon)}{p_2^3 - (p_1^3 - \varepsilon)} \left/ \left(1 - F_{x_1}(p_2^3)\right)\right. \)

When \( \varepsilon \to 0 \), We get the sufficient condition of that arbitrary rationing is profitable.
Reference:


