Journal of Algebra 405 (2014) 168–178

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Examples of clean commutative group rings

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ARTICLE INFO

Article history:
Received 21 July 2009
Available online 4 March 2014
Communicated by Steven Dale Cutkosky

MSC:
primary 16S34
secondary 13F99

Keywords:
Clean ring
Group ring

ABSTRACT

A ring R is said to be clean if each element of R can be written as the sum of a unit and an idempotent. In this article we give examples of clean commutative group rings. In particular, we characterize when the group ring $Z_{(p)}[C_n]$ is clean. The notion of a group ring being clean locally is defined, and it is investigated when the commutative group ring $Z_{(p)}[C_n]$ is clean locally. It is proved that when R is a commutative Hensel ring, the commutative group ring $R[G]$ is clean if and only if G is torsion.

1. Introduction and preliminaries

All rings are associative with identity $1 \neq 0$.

A ring R is said to be *clean* if each element of R can be written as the sum of a unit and an idempotent. This definition is due to Nicholson [12]. Over the last decade there has been a resurgence of interest in the class of clean rings. For some fundamental properties about clean rings as well as a nice history of clean rings we urge the interested reader to check the article [10].

The class of clean rings is extensive. This class includes the classes of commutative zero-dimensional rings, von Neumann regular rings, local rings, and semi-perfect rings.

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http://dx.doi.org/10.1016/j.jalgebra.2014.01.030

By a *local ring* we mean a ring with a unique maximal left ideal; local rings are precisely the indecomposable clean rings. Recall that a ring R with Jacobson radical $\mathfrak{J}(R)$ is called a *semiperfect ring* if idempotents lift modulo $\mathfrak{J}(R)$ and $R/\mathfrak{J}(R)$ is artinian. The following result clarifies the relationship between semiperfect rings and clean rings; it is Theorem 9 of [2]. This theorem will play a pivotal role in our investigations.

Theorem 1.1. *The ring R is semiperfect if and only if it is clean and contains no infinite family of orthogonal idempotents.*

In this article, we aim to shed light on the question of when a commutative group ring is clean. Recall that for a ring with identity, say R, and a group $(G, \cdot, 1)$, the group ring of G over R is the ring $R[G]$ of all formal sums

$$\lambda = \sum_{g \in G} \lambda_g g$$

where $\lambda_i \in R$ and the support of λ, $\operatorname{supp}(\lambda) = \{g \in G\colon \ \lambda_g \neq 0\}$ is finite. Addition is defined pointwise and multiplication is defined by the following: for $\lambda, \mu \in R[G]$

$$\lambda\mu = \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

For more information on the group ring we suggest [5] as a reference. We let C_n denote the multiplicative cyclic group of order n. Since a homomorphic image of clean ring is a clean ring it follows that it is necessary that R is clean whenever $R[G]$ is.

In the next section we characterize when a group ring of the form $\mathbb{Z}_{(p)}[C_n]$ is clean. (Here $\mathbb{Z}_{(p)}$ denotes the localization of the integers at the prime p.) It was shown in [13] that the group ring $\mathbb{Z}_{(7)}[C_3]$ is not semiperfect; this example has been used repeatedly. Later, in [7], it was demonstrated that this same ring is not clean. It then follows that since $\mathbb{Z}_{(p)}$ is a clean ring (it is local) that R being a commutative clean ring is not sufficient for $R[G]$ to be a clean ring. This example provides our motivation, and seems to have motivated some other related results. Ye [15] has shown that the group ring $Z_{(p)}[C_3]$ is semiclean for any prime p, while it was shown by Xiao and Tong in [14] that $Z_{(p)}[C_3]$ is 2-clean for any prime $p \neq 2$. McGovern [11] proved that it is necessary for a commutative group ring $R[G]$ to be a clean ring it is also necessary that G be a torsion group. Moreover, it was also shown that for any elementary 2-group G and commutative ring R, $R[G]$ is clean if and only if R is clean. In [3] the authors have proved that for any $k \geqslant 2$, $\mathbb{Z}_{(5)}[C_{2^k}]$ is not clean, and that for any $n > 2$, there is a prime q such that $\mathbb{Z}_{(q)}[C_n]$ is not clean. They also reprove that if the commutative group ring $R[G]$ is clean, then G must be torsion. The proof in [11] appears to be simpler. (The interested reader is encouraged to see these papers for the appropriate definitions.) We prove the strong result that the group ring $Z_{(p)}[C_3]$ is clean if and only if $p \neq 1$ modulo 3. More generally, we give a complete characterization of when $Z_{(p)}[C_n]$ is clean. The above results will follow nicely from our characterization.

2. The group ring $Z_{(p)}[C_n]$

Proposition 2.1. *Let R be a noetherian ring, and let G be a finitely generated abelian group. The group ring $R[G]$ is clean if and only if it is semiperfect.*

Proof. When G is an abelian group and R is a ring, the group ring $R[G]$ is noetherian if and only if R is noetherian and G is finitely generated by Theorem 2 of Connell [4]. Since noetherian rings cannot contain an infinite family of orthogonal idempotents, the result follows from Theorem 1.1. □

Corollary 2.2. *Let p be a prime, and let n be a positive integer. Then the group ring $\mathbb{Z}_{(p)}[C_n]$ is clean if and only if it is semiperfect.*

Remark 2.3. The statement of the above corollary is pointed out in [3] in the paragraph prior to Example 3.3.

Let p be either a prime or zero. The p-primary component of an abelian group G is the subgroup consisting of all elements whose order is a power of p. When $p = 0$, the p-primary component of G will mean the trivial subgroup. The exponent of a multiplicative group G is the least positive integer n, if one exists, such that $g^n = 1$ for each element $g \in G$. The following result is Theorem 5.8 of [13].

Theorem 2.4. *Let R be a commutative local ring with $\operatorname{char} R/\mathfrak{J}(R) = p \geqslant 0$, and let G be an abelian group with p-primary component G_p. Then the group ring $R[G]$ is semiperfect if and only if $G = G_p$ is a finite group of exponent n and each monic factor of $X^n - 1 \in R/\mathfrak{J}(R)[X]$ can be lifted to a monic factor of $X^n - 1 \in R[X]$.*

This theorem was used by Woods to show that $\mathbb{Z}_{(7)}[C_3]$ is not semiperfect. For any prime p, let $\mathbb{F}_p$ denote the finite field of order p. Notice that $X^3 - 1$ factors as $(X - 1) \times (X - 2)(X - 4)$ over $\mathbb{F}_7$ while it factors as $(X - 1)(X^2 + X + 1)$ over $\mathbb{Z}_{(7)}$. Since $(X^2 + X + 1)$ is irreducible over $\mathbb{Z}_{(7)}$, it follows from Theorem 2.4 that the group ring $\mathbb{Z}_{(7)}[C_3]$ is not semiperfect. In particular, $\mathbb{Z}_{(7)}[C_3]$ is not clean by Corollary 2.2. This result was first shown by Han and Nicholson [7] and was confirmed by Ye [15], who showed that when g is a generator of C_3, the element $2 + 3g \in \mathbb{Z}_{(7)}[C_3]$ cannot be written as the sum of a unit and an idempotent. The following proposition is a consequence of Theorem 2.4 and Corollary 2.2.

Proposition 2.5. *Let p be a prime, let n be a positive integer, and write $n = p^k m$ where $p \nmid m$. Then the group ring $\mathbb{Z}_{(p)}[C_n]$ is clean if and only if each monic factor of $X^m - 1 \in \mathbb{F}_p[X]$ can be lifted to a monic factor of $X^m - 1 \in \mathbb{Z}_{(p)}[X]$.*

Let p be any prime. It is immediate from Proposition 2.5 that the group ring $Z_{(p)}[C_{p^k}]$ is clean for any nonnegative integer k. Even more can be said about these group rings. By

Theorem 19.1 of [5], a group ring $R[G]$ is local if and only if R is local with char $R/\mathfrak{J}(R) = p \neq 0$ and the order of each element of G is a power of p. It follows that the group ring $Z_{(p)}[C_n]$ is local if and only if $n = p^k$ for some nonnegative integer k.

It also follows easily from Proposition 2.5 that the group ring $Z_{(p)}[C_{2p^k}]$ is clean for any nonnegative integer k. When p is an odd prime, simply notice that $X^2 - 1$ factors as $(X-1)(X+1)$ in both $\mathbb{F}_p[X]$ and $\mathbb{Z}_{(p)}[X]$. In particular, this shows that $Z_{(p)}[C_n]$ can be clean without being local.

We can improve upon Proposition 2.5 by looking at the irreducible monic factors of the polynomial $X^m - 1$ in the polynomial rings $\mathbb{F}_p[X]$ and $\mathbb{Z}_{(p)}[X]$, where $p \nmid m$. It will be helpful to recall several well known facts about cyclotomic polynomials. These results can be found, for example, in [9].

Let ζ be a primitive mth root of unity over some field K of characteristic $p \geqslant 0$ where $p \nmid m$. The mth-cyclotomic polynomial over K, denoted $\Phi_m(X)$, is defined to be the polynomial whose roots are the primitive mth-roots of unity over K. Thus, letting $u(m) = \{a \in \mathbb{N}\colon\ 1 \leqslant a \leqslant m,\ gcd(a,m) = 1\}$,

$$\Phi_m(X) = \prod_{a \in u(m)} \left(X - \zeta^a\right).$$

Let ϕ denote the Euler ϕ-function, and notice that $\Phi_m(X)$ is a monic polynomial of degree $\phi(m)$.

It is well known that the coefficients of $\Phi_m(X)$ belong to the prime subfield of K and are integers if the prime subfield of K is the rational numbers. In particular, each cyclotomic polynomial over $\mathbb{Q}$ has integer coefficients, and hence may be viewed as an element of $\mathbb{Z}_{(p)}[X]$.

Let μ denote the Möbius function. It is also well known that the mth-cyclotomic polynomial over K can be found explicitly by the formula

$$\Phi_m(X) = \prod_{d \mid m} \left(X^d - 1\right)^{\mu(m/d)}$$

where the product is taken over all positive divisors d of m. Notice that this formula does not depend on K. In particular, it follows that the mth-cyclotomic polynomial over $\mathbb{F}_p$ is congruent modulo p to the mth-cyclotomic polynomial over $\mathbb{Q}$.

It is an easy observation that the roots of the polynomial $X^m - 1 \in K[X]$ are precisely the mth-roots of unity over K. Since each mth-root of unity is a primitive dth-root of unity for exactly one positive divisor d of m, it follows that

$$X^m - 1 = \prod_{d \mid m} \Phi_d(X)$$

where, once again, the product is taken over all positive divisors d of m. Thus $\Phi_d(X)$ is a monic factor of $X^m - 1$ whenever d divides m. In general, these factors of $X^m - 1$ might be reducible. However, it is well known that the dth-cyclotomic polynomial over $\mathbb{Q}$ is irreducible over $\mathbb{Q}$ for any positive integer d, and hence these polynomials are irreducible over $\mathbb{Z}_{(p)}$.

Given any positive integers m and n with $gcd(m,n) = 1$, we let $\operatorname{ord}_n m$ denote the multiplicative order of m modulo n. Thus $\operatorname{ord}_n m$ is the least positive integer l that satisfies the congruence $m^l \equiv 1$ modulo n. If $\operatorname{ord}_n m = \phi(n)$, then m is said to be a *primitive root of* n. It is well known that the dth-cyclotomic polynomial over $\mathbb{F}_p$ is irreducible over $\mathbb{F}_p$ if and only if p is a primitive root of d. We can now prove the following lemma.

Lemma 2.6. *Let p be any prime, and let m be a positive integer such that $p \nmid m$. Each monic factor of $X^m - 1$ over $\mathbb{F}_p$ can be lifted to a monic factor of $X^m - 1$ over $\mathbb{Z}_{(p)}$ if and only if the dth-cyclotomic polynomial over $\mathbb{F}_p$ is irreducible over $\mathbb{F}_p$ for each positive divisor d of m. That is, if and only if p is a primitive root of d for each positive divisor d of m.*

Proof. Let $\Phi_d(X)$ denote the dth-cyclotomic polynomial over $\mathbb{Q}$, and view this polynomial as an element of $\mathbb{Z}_{(p)}[X]$. Let $\overline{\Phi}_d(X)$ denote the image of $\Phi_d(X)$ in the polynomial ring $\mathbb{F}_p[X]$. It is noted above that the dth-cyclotomic polynomial over $\mathbb{F}_p$ is congruent to $\Phi_d(X)$ modulo p. Since the coefficients of $\Phi_d(X)$ are integers, it follows that $\overline{\Phi}_d(X)$ is precisely the dth-cyclotomic polynomial over $\mathbb{F}_p$.

View $\overline{\Phi}_d(X)$ as the dth-cyclotomic polynomial over $\mathbb{F}_p$. We have shown that $\overline{\Phi}_d(X)$ lifts to $\Phi_d(X)$ for any positive integer d. This implies that $\overline{\Phi}_d(X)$ lifts to a monic factor of $X^m - 1$ in $\mathbb{Z}_{(p)}[X]$ for any positive divisor d of m. It is useful to note that each monic factor of $X^m - 1$ in $\mathbb{F}_p[X]$ can be lifted to a monic factor of $X^m - 1$ in $\mathbb{Z}_{(p)}[X]$ if and only if each irreducible monic factor can be lifted.

If the polynomial $\overline{\Phi}_d(X)$ is irreducible over $\mathbb{F}_p$ for each positive divisor d of m, then these are precisely the irreducible factors of $X^m - 1$ in $\mathbb{F}_p[X]$. Since each of these polynomials can be lifted to a monic factor of $X^m - 1$ in $\mathbb{Z}_{(p)}[X]$, it follows that each monic factor of $X^m - 1$ in $\mathbb{F}_p[X]$ can be lifted. However, if the polynomial $\overline{\Phi}_d(X)$ is reducible for some positive divisor d of m, then $X^m - 1$ will have more irreducible factors in $\mathbb{F}_p[X]$ than in $\mathbb{Z}_{(p)}[X]$. In this case, not all monic factors of $X^m - 1$ in $\mathbb{F}_p[X]$ can be lifted. □

From Proposition 2.5 and Lemma 2.6 we obtain the following necessary and sufficient condition for the group ring $\mathbb{Z}_{(p)}[C_n]$ to be clean.

Proposition 2.7. *Let p be a prime, let n be a positive integer, and write $n = p^k m$ where $p \nmid m$. Then the group ring $\mathbb{Z}_{(p)}[C_n]$ is clean if and only if p is a primitive root of d for each positive divisor d of m.*

This criterion is certainly easier to check than the one given in Proposition 2.5. For instance, to prove that the group ring $\mathbb{Z}_{(7)}[C_3]$ is not clean, simply note that $\operatorname{ord}_3 7 = 1$

while $\phi(3) = 2$, and thus 7 is not a primitive root of 3. This example leads us to the following corollary of Proposition 2.7.

Corollary 2.8. *Let p be any prime. Then the group ring $\mathbb{Z}_{(p)}[C_3]$ is clean if and only if $p \neq 1$ modulo 3.*

To improve upon the result of Proposition 2.7, we need to recall some basic facts about primitive roots from introductory number theory. First, a positive integer n has a primitive root if and only if n is 1, 2, 4, q^l, or $2q^l$ for some odd prime q and positive integer l. This result, sometimes called the Primitive Root Theorem, dates back to at least 1801 when it was proved by Gauss. For a simple algebraic proof of this number theoretic result, see [6].

The next proposition is a consequence of Proposition 2.7 and the Primitive Root Theorem. This result gives a necessary condition for the group ring $Z_{(p)}[C_n]$ to be clean and leads to immediate examples of group rings that are not clean.

Proposition 2.9. *Let p be a prime, let n be a positive integer, and write $n = p^k m$ where $p \nmid m$. If the group ring $\mathbb{Z}_{(p)}[C_n]$ is clean, then m must be 1, 2, 4, q^l, or $2q^l$ for some odd prime q and positive integer l.*

Certainly this is not a sufficient condition for the group ring $\mathbb{Z}_{(p)}[C_n]$ to be clean. In particular, notice that our chief example $\mathbb{Z}_{(7)}[C_3]$ of a group ring that is not clean actually satisfies this condition.

The following facts about primitive roots lead to a necessary and sufficient condition for the group ring $\mathbb{Z}_{(p)}[C_n]$ to be clean. These results can be found in most introductory number theory textbooks. Let q be an odd prime, and let l be any nonnegative integer. If r is a primitive root of q^2, then r is a primitive root of q^l for all l. If r is an odd primitive root of q^l, then r is a primitive root of $2q^l$. These facts and the Primitive Root Theorem give us the following lemma.

Lemma 2.10. *Let p be a prime, let n be a positive integer, and write $n = p^k m$ where $p \nmid m$. Then p is a primitive root of d for each positive divisor d of m if and only if at least one the following conditions is satisfied:*

(1) *$m = 1$ or 2.*
(2) *$m = 4$ and p is a primitive root of 4.*
(3) *$m = q$ or $2q$ for some odd prime q, and p is a primitive root of q.*
(4) *$m = q^l$ or $2q^l$ for some odd prime q, and p is a primitive root of q^2.*

As an immediate consequence of Proposition 2.7 and Lemma 2.10, we have the following characterization of when the group ring $Z_{(p)}[C_n]$ is clean.

Theorem 2.11. *Let p be a prime, let n be a positive integer, and write $n = p^k m$ where $p \nmid m$. Then the group ring $\mathbb{Z}_{(p)}[C_n]$ is clean if and only if at least one of the following conditions is satisfied:*

(1) $m = 1$ *or* 2.
(2) $m = 4$ *and p is a primitive root of* 4.
(3) $m = q$ *or* $2q$ *for some odd prime q, and p is a primitive root of q.*
(4) $m = q^l$ *or* $2q^l$ *for some odd prime q, and p is a primitive root of q^2.*

3. The group ring $\mathbb{Z}_{(p)}[G]$

In this section we investigate the cleanliness of the group ring $\mathbb{Z}_{(p)}[G]$, where G is any abelian group. For any group ring $R[G]$, let us say that $R[G]$ is *clean locally* (*semiperfect locally*) when $R[H]$ is clean (semiperfect) for each finitely generated subgroup H of G. The main result of this section is a characterization of when the group ring $\mathbb{Z}_{(p)}[G]$ is clean locally. We begin by showing that when a group ring is clean locally, it is also clean. This is mentioned in [3] in the paragraph after Proposition 3.7.

Theorem 3.1. *If the group ring $R[G]$ is clean locally, then it is clean.*

Proof. Given any element λ of $R[G]$, let G_λ denote the support group $\langle \mathrm{supp}(\lambda)\rangle$. Then G_λ is a finitely generated subgroup of G, and hence the group ring $R[G_\lambda]$ is clean. Furthermore, $R[G_\lambda]$ is a clean subring of $R[G]$ containing λ. Thus, λ can be written as a sum of a unit and an idempotent in $R[G]$, and the result follows. □

The following is a corollary to Proposition 2.1.

Corollary 3.2. *Let R be a noetherian ring, and let G be an abelian group. Then $R[G]$ is clean locally if and only if it is semiperfect locally. In particular, for any prime p, the group ring $\mathbb{Z}_{(p)}[G]$ is clean locally if and only if it is semiperfect locally.*

For the moment, we address the question of when the commutative group ring $\mathbb{Z}_{(p)}[G]$ is semiperfect. We actually give three characterizations of when $\mathbb{Z}_{(p)}[G]$ is semiperfect. The first is a consequence of Theorem 2.4 and Lemma 2.6.

Theorem 3.3. *Let p be any prime, and let G be an abelian group with p-primary component G_p. Then the group ring $\mathbb{Z}_{(p)}[G]$ is semiperfect if and only if G/G_p is a finite group of exponent n and p is a primitive root of d for each positive divisor d of n.*

The next result follows from Theorem 3.3 and Lemma 2.10.

Theorem 3.4. *Let p be any prime, and let G be an abelian group with p-primary component G_p. Then the group ring $\mathbb{Z}_{(p)}[G]$ is semiperfect if and only if G/G_p is a finite group of exponent n and at least one of the following conditions is satisfied*:

(1) $n = 1$ *or* 2.
(2) $n = 4$ *and p is a primitive root of* 4.
(3) $n = q$ *or* $2q$ *for some odd prime q, and p is a primitive root of q.*
(4) $n = q^l$ *or* $2q^l$ *for some odd prime q, and p is a primitive root of q^2.*

Let p be any prime. A group G is called a *p-group* if the order of each element of G is a power of p. An abelian group G is called an *elementary* (*abelian*) *p-group* if each element of G has order p. The next result is a corollary to Theorem 3.4.

Theorem 3.5. *Let p be any prime, and let G be an abelian group with p-primary component G_p. Then the group ring $\mathbb{Z}_{(p)}[G]$ is semiperfect if and only if G/G_p is finite and at least one of the following conditions is satisfied.*

(1) *$G/G_p = 1$, i.e., G is a p-group.*
(2) *G/G_p is an elementary 2-group.*
(3) *G/G_p is a 2-group of exponent 4, and p is a primitive root of 4.*
(4) *G/G_p is an elementary q-group for some odd prime q, and p is a primitive root of q.*
(5) *G/G_p is a q-group for some odd prime q, and p is a primitive root of q^2.*
(6) *G/G_p is isomorphic to the direct product of an elementary 2-group and an elementary q-group for some odd prime q, and p is a primitive root of q.*
(7) *G/G_p is isomorphic to the direct product of an elementary 2-group and a q-group for some odd prime q, and p is a primitive root of q^2.*

We now characterize when the commutative group ring $\mathbb{Z}_{(p)}[G]$ is clean locally.

Theorem 3.6. *Let p be any prime, and let G be an abelian group with p-primary component G_p. Then the group ring $\mathbb{Z}_{(p)}[G]$ is clean locally if and only if at least one of the following conditions is satisfied.*

(1) *$G/G_p = 1$, i.e., G is a p-group.*
(2) *G/G_p is an elementary 2-group.*
(3) *G/G_p is a 2-group of exponent 4, and p is a primitive root of 4.*
(4) *G/G_p is an elementary q-group for some odd prime q, and p is a primitive root of q.*
(5) *G/G_p is a q-group for some odd prime q, and p is a primitive root of q^2.*
(6) *G/G_p is isomorphic to the direct product of an elementary 2-group and an elementary q-group for some odd prime q, and p is a primitive root of q.*
(7) *G/G_p is isomorphic to the direct product of an elementary 2-group and a q-group for some odd prime q, and p is a primitive root of q^2.*

Proof. Let G be any abelian group that satisfies one of the conditions (1)–(7) of the theorem, and notice that any finitely generated subgroup of G will also satisfy one of these conditions. Moreover, notice that G is a torsion group, and thus any finitely generated subgroup of G is finite. It follows from Theorem 3.5 that $\mathbb{Z}_{(p)}[G]$ is semiperfect locally, and hence $\mathbb{Z}_{(p)}[G]$ is clean locally by Corollary 3.2. For the other direction, we prove the contrapositive. Let G be any abelian group that does not satisfy any of the conditions (1)–(7) of the theorem. Then there exists a finitely generated subgroup H of G that does not satisfy any of these conditions. It follows from Theorem 3.5 that $\mathbb{Z}_{(p)}[G]$ is not semiperfect locally, and hence not clean locally. □

As a consequence of Theorem 3.5 and Theorem 3.6, the commutative group ring $\mathbb{Z}_{(p)}[G]$ is semiperfect if and only if it is clean locally and G/G_p is finite. In particular, we have two immediate examples of group rings that are clean locally but not semiperfect. First, suppose that p and q are distinct primes with p a primitive root of q. If G is any infinite elementary abelian q-group, then the group ring $\mathbb{Z}_{(p)}[G]$ is clean locally but not semiperfect. Second, suppose that p and q are distinct primes with p a primitive root of q^2. If G is the group of all q-power roots of unity in $\mathbb{C}$, then the group ring $\mathbb{Z}_{(p)}[G]$ is clean locally but not semiperfect.

Remark 3.7. By what we know now the group ring $\mathbb{Z}_{(7)}[S_3]$ is a clean ring which is not clean locally. We do not know whether there is such an example involving a commutative group ring.

4. Group rings over a Hensel ring

In this final section we investigate the cleanliness of the group ring $R[G]$, where R is a commutative Hensel ring and G is any abelian group. Let R be a local ring, and for any polynomial $f \in R[X]$, let $\overline{f}$ denote the image of f in the ring $R/\mathfrak{J}(R)[X]$. Following Azumaya [1], we say that the *Hensel Lemma* holds for f if for any relatively prime polynomials $g_0, h_0 \in R/\mathfrak{J}(R)[X]$ with g_0 monic and $g_0h_0 = \overline{f}$, there exist polynomials $g, h \in R[X]$ with g monic such that $\bar{g} = g_0$, $\overline{h} = h_0$, and $gh = f$. A local ring R is called a *Hensel ring* if the Hensel Lemma holds for each monic polynomial in $R[X]$. For example, the ring $\mathbb{Z}_p$ of p-adic integers is a Hensel ring. We prove that when R is a commutative Hensel ring and G is an abelian group, the group ring $R[G]$ is clean if and only if G is torsion.

We begin with a lemma that makes our interest in Hensel rings clear.

Lemma 4.1. *Let R be a commutative Hensel ring with* $\operatorname{char} R/\mathfrak{J}(R) = p \geqslant 0$ *and let m be any positive integer such that $p \nmid m$. Then each monic factor of $X^m - 1$ in $R/\mathfrak{J}(R)[X]$ can be lifted to a monic factor of $X^m - 1$ in $R[X]$.*

Proof. Since the characteristic of $R/\mathfrak{J}(R)$ does not divide m, the mth-roots of unity over $R/\mathfrak{J}(R)$ are distinct. In particular, this tells us that any two polynomials in $R/\mathfrak{J}(R)[X]$

whose product is $X^m - 1$ must be relatively prime. Since R is a Hensel ring, the Hensel Lemma holds for $X^m - 1$, and the result follows. □

As an immediate consequence of Theorem 2.4 and Lemma 4.1, we have the following proposition, which characterizes when the group ring of an abelian group over a commutative Hensel ring is semiperfect.

Proposition 4.2. *Let R be a commutative Hensel ring with* $\operatorname{char} R/\mathfrak{J}(R) = p \geqslant 0$, *and let G be an abelian group with p-primary component G_p. Then the group ring $R[G]$ is semiperfect if and only if G/G_p is finite.*

We are now ready to prove the main result of this section.

Theorem 4.3. *Let R be a commutative Hensel ring, and let G be any abelian group. Then the following are equivalent*:

(1) *$R[G]$ is semiperfect locally,*
(2) *$R[G]$ is clean locally,*
(3) *$R[G]$ is clean,*
(4) *G is torsion.*

Proof. Notice that (1) implies (2) by Theorem 1.1, (2) implies (3) by Theorem 3.1, and (3) implies (4) has already been pointed out in the first section. Thus it suffices to show that (4) implies (1). Let G be a torsion group, and let H be any finitely generated subgroup of G. Since any finitely generated subgroup of a torsion group is finite, H is finite, and hence $R[H]$ is semiperfect by Proposition 4.2. Therefore $R[G]$ is semiperfect locally. □

Acknowledgment

This article is part of the author's dissertation [8] and is credited solely to the first author. Correspondence should be directed towards the author's Ph.D. adviser Warren Wm. McGovern, H.L. Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458; warren.mcgovern@fau.edu.

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